

FINITE PLASTICITY IN $\mathbf{P}^\top \mathbf{P}$

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ABSTRACT. We discuss a finite-plasticity model based on the symmetric tensor $\mathbf{P}^\top \mathbf{P}$ instead of the classical plastic strain \mathbf{P} . Such a model structure arises from assuming that the material behavior is invariant with respect to frame transformations of the intermediate configuration. The resulting variational model is lower-dimensional, symmetric, and based solely on the reference configuration. We discuss the existence of energetic solutions both at the material-point level and for the quasistatic boundary-value problem. These solutions are constructed as limits of time discretizations. Eventually, the linearization of the model for small deformations is ascertained via a rigorous evolutive- Γ -convergence argument.

1. INTRODUCTION

The inelastic behavior of polycrystalline solids is classically described in terms of the deformation gradient \mathbf{F} with respect to the reference configuration [29]. As the elastic response is observed to be largely independent from prior plastic distortion of the crystalline structure, the deformation gradient is usually decomposed into an elastic and a plastic part. While this decomposition is additive in the small-deformation regime, at finite strains, a multiplicative decomposition $\mathbf{F} = \mathbf{F}_e \mathbf{P}$ is used instead [32, 34]. Here \mathbf{F}_e is the elastic deformation tensor, describing indeed the elastic response of the medium, and \mathbf{P} is the plastic deformation tensor, encoding the information on the plastic state. Although other options have been advanced, see for instance [10, 16, 37, 55], the multiplicative decomposition has now turned to be the reference in finite plasticity. A justification for this decomposition has been recently provided in [60, 61] on the basis micromechanical considerations.

Based on the multiplicative decomposition, the elastoplastic evolution of the medium is described by the time evolution of \mathbf{F}_e and \mathbf{P} . This results from the competition between energy-storage and plastic-dissipation mechanisms [56, 65]. A first structural restriction to the modeling choice is that of *frame indifference* [29], imposing indeed the elastic state of the material to be completely represented in terms of the so-called (*right*) *Cauchy-Green tensor* $\mathbf{F}_e^\top \mathbf{F}_e$. Moving from by this observation, the focus of this paper is to address the possibility of formulating and analyzing finite plasticity in terms of the corresponding *plastic* Cauchy-Green tensor $\mathbf{P}^\top \mathbf{P}$ instead of \mathbf{P} .

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Finite-plasticity models in terms of $\mathbf{P}^\top \mathbf{P}$ bear significant advantages with respect to formulations in \mathbf{P} . At first, variables are symmetric and positive definite, reducing indeed the degrees of freedom of the problem. Furthermore, \mathbf{C}_p being symmetric plays a significant computational role allowing the use of efficient algorithms, especially in connection with power- and exponential-matrix evaluations [18]. Secondly, a model in $\mathbf{P}^\top \mathbf{P}$ is fully defined on the reference configuration of the medium, avoiding the necessity of any intermediate configuration, a commonly controversial issue [53]. Moreover, such a fully Lagrangian formulation is better adapted to finite-element approximations, which are then to be defined for all variables on the fixed reference configuration. Triggered by these appealing features, finite-plasticity models based on $\mathbf{P}^\top \mathbf{P}$ have already attracted attention [64, 36, 40, 70]. The reader is referred to the recent [54] where a comparative study of these as well as the current model is provided. In the context of shape memory materials, a model based on $\mathbf{P}^\top \mathbf{P}$ is advanced in [20, 21] and variationally reformulated and analyzed in [26].

The aim of this paper is to provide a comprehensive discussion of finite plasticity expressed in terms of $\mathbf{P}^\top \mathbf{P}$, both at the modeling and at the analytical level. Starting from classical associative finite plasticity in terms of \mathbf{P} , we discuss a general frame allowing an equivalent reformulation in $\mathbf{P}^\top \mathbf{P}$. This relies on a quite natural *plastic-invariance assumption*, translating indeed the indifference of the model with respect to rotations of the intermediate configuration. Quite remarkably, the model in $\mathbf{P}^\top \mathbf{P}$ turns out to be associative with respect to the new variables as well.

The variational structure of the model allows us to prove the existence of variational solutions of *energetic type* [22, 42, 51]. At the material-point level we obtain such an existence under very general assumptions on the model ingredients, in particular on the coercivity of the energy. We then turn to the analysis of the quasistatic-evolution setting resulting from the combination of the constitutive material relation with the equilibrium system. Also at the quasistatic evolution level energetic solutions are proved to exist, provided the energy is polyconvex [4] and augmented by a gradient term of the form $|\nabla(\mathbf{P}^\top \mathbf{P})|$. Such a term describes nonlocal plastic effects and is inspired to the by-now classical *gradient plasticity* theory [23, 24, 52]. In particular, its occurrence turns out to be crucial in order to prevent the formation of plastic microstructures and ultimately ensures the necessary compactness for the analysis. In the context of the mathematical analysis of finite plasticity, the introduction of suitable regularizing terms on the plastic variables seems at the moment unavoidable. The *incremental* problem has been tackled under the regularizing effect of a term in $\text{curl } \mathbf{P}$ in [44]. An existence result in finite (incremental) plasticity without gradient terms is in [41] where however substantial restrictions on modeling choices are imposed. The only other available quasistatic-evolution existence result is for the formulation in \mathbf{P} [38] and features a regularizing term in $\nabla \mathbf{P}$ as well.

The existence results, both at the material-point and the quasistatic-evolution level, results from passing to the limit in implicit time-discretization schemes. As a by-product we hence devise the convergence of such schemes both in terms of solution trajectories and of energy and dissipation.

A second important focus of our analysis is the rigorous justification of the classical linearization approach for small deformations. Within the small-deformation regime it is indeed customary to leave the nonlinear finite-strain frame and resort to linearized theories. This model reduction is classically justified by heuristic Taylor-expansion arguments. Here, we aim instead at providing a rigorous linearization proof by means of an *evolutionary* Γ -convergence analysis in the spirit of the general abstract theory of [49]. This rigorous limiting procedure is devised both at the material-point and at the quasistatic-evolution level. Note that a rigorous convergence result in case of the \mathbf{P} -based formulation was provided in [50]. Our results extends that of [50] to the case of $\mathbf{P}^\top \mathbf{P}$ -plasticity. In contrast with [50] we discuss here the convergence of solutions for which existence is known. This involves the additional difficulty of discussing the convergence of the gradient terms.

We describe the constitutive model and the role of plastic-rotation indifference on Section 2. Section 3 provides a minimal toolbox on energetic formulations of rate-independent systems and the corresponding approximation via evolutionary Γ -convergence. The existence of energetic solutions of the constitutive model at the material-point level is discussed in Section 4 and the corresponding small-deformation limit is presented in Section 5. The quasistatic-evolution problem is introduced in Section 6. The corresponding existence result is then presented in Section 7 whereas Section 8 contains the detail of the small-deformation limit.

2. CONSTITUTIVE MODEL

We devote this section to the specification of the finite-plasticity model in study. As already commented in the Introduction, this corresponds to classical associative finite plasticity under an invariance assumption with respect to plastic rotations. We limit ourselves at introducing the constitutive relation, referring indeed the reader to the monographs [30, 56, 65] for additional material and detail on finite-plasticity formulations.

Before going on let us record here that finite plasticity is to-date a still controversial subject [53]. It is not our intention to contribute new mechanical arguments to the ongoing discussion. On the contrary our aim is to present the possibly simplest model in \mathbf{C}_p showing a sound variational structure. The specific form of our constitutive model seems to be new [54]. Still, we believe that the main interest in this rather simplified case relies on the quite detailed mathematical analysis that such a variational structure allows.

2.1. Tensors. We focus on the three-dimensional setting and systematically use boldface symbols in order to indicate 2-tensors in \mathbb{R}^3 . The corresponding space is denoted by $\mathbb{R}^{3 \times 3}$. Given $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ we classically define its trace as $\text{tr } \mathbf{A} := A_{ii}$ (summation convention), its deviatoric part as $\text{dev } \mathbf{A} = \mathbf{A} - (\text{tr } \mathbf{A})\mathbf{I}/3$ where \mathbf{I} is the identity 2-tensor, and its (Frobenius) norm as $|\mathbf{A}|^2 := \text{tr } (\mathbf{A}^\top \mathbf{A})$ where the symbol \top denotes transposition. The contraction product between 2-tensors is $\mathbf{A}:\mathbf{B} := A_{ij}B_{ij}$ and we classically denote the scalar product of vectors in \mathbb{R}^3 by $a \cdot b := a_i b_i$. The symbols $\mathbb{R}_{\text{sym}}^{3 \times 3}$ and $\mathbb{R}_{\text{sym}+}^{3 \times 3}$ stand

for the subsets of $\mathbb{R}^{3 \times 3}$ of symmetric tensors and of symmetric positive-definite tensors, respectively. Moreover, $\mathbb{R}_{\text{dev}}^{3 \times 3}$ indicates the space of symmetric deviatoric tensors, namely $\mathbb{R}_{\text{dev}}^{3 \times 3} := \{\mathbf{A} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \mid \text{tr } \mathbf{A} = 0\}$. We shall use also the following tensor sets

$$\begin{aligned} \text{SL} &:= \{\mathbf{A} \in \mathbb{R}^{3 \times 3} \mid \det \mathbf{A} = 1\}, \\ \text{SO} &:= \{\mathbf{A} \in \text{SL} \mid \mathbf{A}^{-1} = \mathbf{A}^\top\}, \\ \text{GL}^+ &:= \{\mathbf{A} \in \mathbb{R}^{3 \times 3} \mid \det \mathbf{A} > 0\}, \\ \text{GL}_{\text{sym}}^+ &:= \text{GL}^+ \cap \mathbb{R}_{\text{sym}}^{3 \times 3}, \\ \text{SL}_{\text{sym}}^+ &:= \text{SL} \cap \mathbb{R}_{\text{sym}}^{3 \times 3}. \end{aligned}$$

The tensor $\text{cof } \mathbf{A}$ is the *cofactor matrix* of \mathbf{A} . For \mathbf{A} invertible we have that $\text{cof } \mathbf{A} = (\det \mathbf{A}) \mathbf{A}^{-\top}$. For any symmetric positive-definite matrix $\mathbf{A} \in \mathbb{R}_{\text{sym}+}^{3 \times 3}$, the real power \mathbf{A}^s is classically defined, for any $s \in \mathbb{R}$, in terms of its eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i > 0$ and

$$\text{tr } \mathbf{A}^s = \lambda_1^s + \lambda_2^s + \lambda_3^s, \quad \det \mathbf{A}^s = (\lambda_1 \lambda_2 \lambda_3)^s.$$

In particular, the square root $\mathbf{A}^{1/2}$ is uniquely defined in GL_{sym}^+ . The matrix logarithm $\log \mathbf{C}_p$ is globally uniquely defined in SL_{sym}^+ . In particular, one has that $\text{tr}(\log \mathbf{C}_p) = \log(\det \mathbf{C}_p) = 0$ for all $\mathbf{C}_p \in \text{SL}_{\text{sym}}^+$. Given any symmetric, positive-definite 4-tensor \mathbb{C} we denote by $|\mathbf{A}|_{\mathbb{C}}^2 := \mathbf{A} : \mathbb{C} \mathbf{A}$ the corresponding induced (squared) norm on $\mathbb{R}_{\text{sym}}^{3 \times 3}$. The product $\mathbb{C} \mathbf{A}$ is here classically defined as $(\mathbb{C} \mathbf{A})_{ij} := \mathbb{C}_{ijkl} A_{lk}$. For all 3-tensors \mathbf{D} (arising in this context as gradients of 2-tensor fields) we define $|\mathbf{D}|^2 := D_{ijk}^2$, the partial trasposition $(\mathbf{D}^\top)_{ijk} = D_{jik}$, and the product with the 2-tensor \mathbf{A} as $(\mathbf{D} \mathbf{A})_{ijk} = D_{ijl} A_{lk}$ and $(\mathbf{A} \mathbf{D})_{ijk} = A_{il} D_{ljk}$. Note that, along with these definitions, $|\mathbf{D} \mathbf{A}|, |\mathbf{A} \mathbf{D}| \leq |\mathbf{A}| |\mathbf{D}|$.

In the following we denote by $\partial\varphi$ the subdifferential of the smooth or of the convex, proper, and lower semicontinuous function $\varphi : E \rightarrow (-\infty, \infty]$ where E is a normed space with dual E^* and duality pairing $\langle \cdot, \cdot \rangle$ [7]. In particular, $y^* \in \partial\varphi(x)$ iff $\varphi(x) < \infty$ and

$$\langle y^*, w - x \rangle \leq \varphi(w) - \varphi(x) \quad \forall w \in E.$$

A *caveat* on notation: in the following we use the same symbol c in order to indicate a generic constant, possibly depending on data and varying from line to line.

2.2. Deformation. We consider an elastoplastic body occupying the reference configuration Ω , which is assumed to be a nonempty, open, connected, and bounded subset of \mathbb{R}^3 with Lipschitz boundary $\partial\Omega$. The three-dimensional setting is here chosen for the sake of notational definiteness only: both modeling and analysis could be reformulated in one or two dimensions.

The deformation of the body is described by $y : \Omega \rightarrow \mathbb{R}^3$ and is assumed to be such that the deformation gradient $\mathbf{F} := \nabla y$ is almost everywhere defined and belongs to GL^+ . The deformation gradient \mathbf{F} is classically decomposed as [32, 34]

$$\mathbf{F} = \mathbf{F}_e \mathbf{P} \tag{2.1}$$

where \mathbf{F}_e denotes the *elastic* part of \mathbf{F} and \mathbf{P} its *plastic* part. In particular, the plastic tensor \mathbf{P} describes the internal plastic state of the material and fulfills

$$\det \mathbf{P} = 1$$

in order to express the *isochoric* nature of plastic deformations, as customary in metal plasticity [65]. The heuristics for the multiplicative decomposition (2.1) resides in the classical chain rule: in case y can be interpreted as a composition $y_e \circ y_p$ of an elastic and a plastic deformation, the set $y_p(\Omega)$ is termed *intermediate* (or *structural*) configuration and $\nabla y = \nabla y_e(y_p) \nabla y_p$. Note nonetheless that the tensors \mathbf{F}_e and \mathbf{P} need not be gradients as the compatibility conditions $\text{curl } \mathbf{F}_e = \mathbf{0}$ and $\text{curl } \mathbf{P} = \mathbf{0}$ may not hold. Correspondingly, the intermediate configuration can be understood in a local sense only [53]. We refer to the recent [60, 61] for a justification of the multiplicative decomposition (2.1) in two dimensions consisting in a kinematic analysis of elastoplastic deformation in plastic-slip and dislocation systems.

The (*right*) *Cauchy-Green* symmetric tensors associated to the three deformation gradients are defined by

$$\mathbf{C} := \mathbf{F}^\top \mathbf{F} \in \text{GL}_{\text{sym}}^+, \quad \mathbf{C}_e := \mathbf{F}_e^\top \mathbf{F}_e \in \text{GL}_{\text{sym}}^+, \quad \mathbf{C}_p := \mathbf{P}^\top \mathbf{P} \in \text{SL}_{\text{sym}}^+.$$

In particular, we have that $\det \mathbf{C}_p = (\det \mathbf{P})^2 = 1$. Note that these tensors are all true tensorial quantities, all defined on the reference configuration, whereas \mathbf{F} , \mathbf{F}_e , \mathbf{P} are two-points tensors.

2.3. Energy. The evolution of the elastoplastic body is governed by the interplay between energy-storage mechanisms and plastic-dissipative effects. We assume from the very beginning the response of the medium to be *hyperelastic* [69] and start by specifying the *energy density* of the medium by imposing the additive decomposition

$$W_e(\mathbf{F}_e) + W_p(\mathbf{P}) \tag{2.2}$$

into an *elastic* and a *plastic* (or *hardening*) energy term.

The elastic energy density $W_e : \text{GL}^+ \rightarrow [0, \infty)$ is required to be C^1 and *frame indifferent* [69], namely

$$W_e(\mathbf{R}\mathbf{F}_e) = W_e(\mathbf{F}_e) \quad \forall \mathbf{R} \in \text{SO}. \tag{2.3}$$

Frame indifference implies that the elastic energy can be expressed solely in terms of the tensor \mathbf{C}_e . Indeed, given $\mathbf{F}_e \in \text{GL}^+$ by polar decomposition there exists a rotation matrix $\mathbf{R} \in \text{SO}$ such that $\mathbf{F}_e = \mathbf{R}\mathbf{C}_e^{1/2}$ and

$$W_e(\mathbf{F}_e) = W_e(\mathbf{R}^\top \mathbf{F}_e) = W_e(\mathbf{C}_e^{1/2}) =: \widehat{W}_e(\mathbf{C}_e)$$

where now $\widehat{W}_e : \text{GL}_{\text{sym}}^+ \rightarrow [0, \infty)$.

We admit here hardening effects of a purely kinematic nature. These are modulated by the *plastic-energy density* $W_p : \text{SL} \rightarrow [0, \infty]$, which we assume to be C^1 on its domain. Let us explicitly remark that we are not considering here additional internal hardening dynamics. In particular, isotropic hardening is not directly included in our frame. Our

choice is motivated by the mere sake of simplicity. Additional internal parameters could be considered as well.

2.4. Plastic-rotation indifference. The crucial assumption of our analysis is that the material behavior is invariant by plastic rotations. This invariance is formulated as

$$W_e(\mathbf{F}_e \mathbf{Q}) = W_e(\mathbf{F}_e), \quad W_p(\mathbf{Q} \mathbf{P}) = W_p(\mathbf{P}) \quad \forall \mathbf{Q} \in \text{SO} \quad (2.4)$$

for all $\mathbf{F}_e \in \text{GL}^+$ and $\mathbf{P} \in \text{SL}$. The condition on W_e corresponds to *isotropy*, whereas W_p can be nonisotropic instead. The condition on W_p is then nothing but frame indifference with respect to the intermediate configuration.

By using the polar decomposition $\mathbf{P} = \mathbf{Q} \mathbf{C}_p^{1/2}$ for $\mathbf{Q} \in \text{SO}$ we have

$$\mathbf{F}_e = \mathbf{F} \mathbf{P}^{-1} = \mathbf{F} \mathbf{C}_p^{-1/2} \mathbf{Q}^\top.$$

The isotropy of W_e from (2.4) then yields $W_e(\mathbf{F}_e) = W_e(\mathbf{F} \mathbf{C}_p^{-1/2})$. By combining frame indifference and isotropy of W_e one can equivalently rewrite the elastic energy density as

$$\begin{aligned} W_e(\mathbf{F}_e) &= W_e(\mathbf{F} \mathbf{P}^{-1}) = W_e(\mathbf{F} \mathbf{C}_p^{-1/2}) = \widehat{W}_e((\mathbf{F} \mathbf{C}_p^{-1/2})^\top \mathbf{F} \mathbf{C}_p^{-1/2}) \\ &= \widehat{W}_e(\mathbf{C}_p^{-1/2} \mathbf{C} \mathbf{C}_p^{-1/2}). \end{aligned}$$

On the other hand, the invariance of W_p under plastic rotations (2.4) entails that $W_p(\mathbf{P}) = W_p(\mathbf{C}_p^{1/2})$. We hence define the function $\widehat{W}_p : \text{SL}_{\text{sym}}^+ \rightarrow [0, \infty]$ by

$$\widehat{W}_p(\mathbf{C}_p) := W_p(\mathbf{C}_p^{1/2})$$

and rewrite the *energy density* (2.2) as

$$\begin{aligned} W(\mathbf{C}, \mathbf{P}) &= \widehat{W}(\mathbf{C}, \mathbf{C}_p) = \widehat{W}_e(\mathbf{P}^{-\top} \mathbf{C} \mathbf{P}^{-1}) + W_p(\mathbf{P}) \\ &= \widehat{W}_e(\mathbf{C}_p^{-1/2} \mathbf{C} \mathbf{C}_p^{-1/2}) + \widehat{W}_p(\mathbf{C}_p). \end{aligned}$$

The state of the system is hence described by the pair

$$(\mathbf{C}, \mathbf{C}_p) \in \text{GL}_{\text{sym}}^+ \times \text{SL}_{\text{sym}}^+.$$

Henceforth, we systematically employ the hat superscript in order to identify quantities written in terms of the Cauchy-Green tensors \mathbf{C}_e and \mathbf{C}_p .

2.5. Constitutive relations. In order to introduce constitutive relations we shall here follow the classical *Coleman-Noll procedure* [11]. Let us assume sufficient smoothness and compute

$$\begin{aligned} \frac{d}{dt} W(\mathbf{C}, \mathbf{P}) &= \partial_{\mathbf{C}} W : \dot{\mathbf{C}} + \partial_{\mathbf{P}} W : \dot{\mathbf{P}} =: \mathbf{S} : \frac{1}{2} \dot{\mathbf{C}} - \mathbf{N} : \dot{\mathbf{P}} \\ &= 2 \mathbf{P}^{-1} \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) \mathbf{P}^{-\top} : \frac{1}{2} \dot{\mathbf{C}} - \left(2 \mathbf{C}_e \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) \mathbf{P}^{-\top} - 2 \mathbf{P} \partial_{\mathbf{C}_p} \widehat{W}_p(\mathbf{C}_p) \right) : \dot{\mathbf{P}} \quad (2.5) \end{aligned}$$

or, by using the variables $(\mathbf{C}, \mathbf{C}_p)$,

$$\begin{aligned} \frac{d}{dt} \widehat{W}(\mathbf{C}, \mathbf{C}_p) &= \partial_{\mathbf{C}} W : \dot{\mathbf{C}} + \partial_{\mathbf{C}_p} W : \dot{\mathbf{C}}_p =: \mathbf{S} : \frac{1}{2} \dot{\mathbf{C}} - \mathbf{T} : \frac{1}{2} \dot{\mathbf{C}}_p \\ &= 2\mathbf{P}^{-1} \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) \mathbf{P}^{-\top} : \frac{1}{2} \dot{\mathbf{C}} - (2\mathbf{P}^{-1} \mathbf{C}_e \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) \mathbf{P}^{-\top} - 2\partial_{\mathbf{C}_p} \widehat{W}_p(\mathbf{C}_p)) : \frac{1}{2} \dot{\mathbf{C}}_p. \end{aligned} \quad (2.6)$$

We have here used the coaxiality of \mathbf{C}_e and $\partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e)$, which is in turn a consequence of the isotropy of W_e from (2.4), as well as the symmetry of $\partial_{\mathbf{C}_p} \widehat{W}_p$. The above computation shows that the classical *second Piola-Kirchhoff stress tensor* \mathbf{S}

$$\mathbf{S} := 2\mathbf{P}^{-1} \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) \mathbf{P}^{-\top} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \quad (2.7)$$

is the conjugate variable to \mathbf{C} whereas the evolution of \mathbf{P} and \mathbf{C}_p is driven by the corresponding conjugated *thermodynamic forces*

$$\begin{aligned} \mathbf{N} &:= -\partial_{\mathbf{P}} W(\mathbf{C}, \mathbf{P}) = 2\mathbf{C}_e \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) \mathbf{P}^{-\top} - 2\mathbf{P} \partial_{\mathbf{C}_p} \widehat{W}_p(\mathbf{C}_p), \\ \mathbf{T} &:= -2\partial_{\mathbf{C}_p} W(\mathbf{C}, \mathbf{C}_p) = 2\mathbf{P}^{-1} \mathbf{C}_e \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) \mathbf{P}^{-\top} - 2\partial_{\mathbf{C}_p} \widehat{W}_p(\mathbf{C}_p), \end{aligned}$$

respectively. The expression of \mathbf{T} follows along the same computations of [26]. In particular, we use the fact that, for all $\mathbf{B} \in \mathbb{R}^{3 \times 3}$,

$$\mathbf{P}^{-\top} \mathbf{B} \mathbf{P}^{-1} = \mathbf{P}^{-\top} (\partial_{\mathbf{C}_p} \mathbf{P}^{\top} : \mathbf{B}) + (\partial_{\mathbf{C}_p} \mathbf{P} : \mathbf{B}) \mathbf{P}^{-1}, \quad (2.8)$$

in order to compute

$$\begin{aligned} \partial_{\mathbf{C}_p} \widehat{W}_e(\mathbf{C}_p^{-1/2} \mathbf{C} \mathbf{C}_p^{-1/2}) : \mathbf{B} &= \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) : \partial_{\mathbf{P}} (\mathbf{P}^{-\top} \mathbf{C} \mathbf{P}^{-1}) : \partial_{\mathbf{C}_p} \mathbf{P} : \mathbf{B} \\ &= -\partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) : \mathbf{P}^{-\top} (\partial_{\mathbf{C}_p} \mathbf{P}^{\top} : \mathbf{B}) \mathbf{P}^{-\top} \mathbf{C} \mathbf{P}^{-1} - \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) : \mathbf{P}^{-\top} \mathbf{C} \mathbf{P}^{-1} (\partial_{\mathbf{C}_p} \mathbf{P} : \mathbf{B}) \mathbf{P}^{-1} \\ &= -\partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) \mathbf{C}_e : \mathbf{P}^{-\top} (\partial_{\mathbf{C}_p} \mathbf{P}^{\top} : \mathbf{B}) - \mathbf{C}_e \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) : (\partial_{\mathbf{C}_p} \mathbf{P} : \mathbf{B}) \mathbf{P}^{-1} \\ &\stackrel{(2.8)}{=} -\mathbf{C}_e \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) : (\mathbf{P}^{-\top} (\partial_{\mathbf{C}_p} \mathbf{P}^{\top} : \mathbf{B}) + (\partial_{\mathbf{C}_p} \mathbf{P} : \mathbf{B}) \mathbf{P}^{-1}) \\ &= -\mathbf{C}_e \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) : \mathbf{P}^{-\top} \mathbf{B} \mathbf{P}^{-1} = -\mathbf{P}^{-1} \mathbf{C}_e \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) \mathbf{P}^{-\top} : \mathbf{B}. \end{aligned}$$

Note that the tensors \mathbf{N} and \mathbf{T} fulfill the basic relation

$$\mathbf{N} = \mathbf{P} \mathbf{T}. \quad (2.9)$$

As the tensors $\mathbf{C}_e \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e)$ and $\partial_{\mathbf{C}_p} \widehat{W}_p(\mathbf{C}_p)$ are symmetric, the tensor \mathbf{T} is symmetric as well.

2.6. Flow rule in terms of \mathbf{P} . The plastic evolution is formulated in terms of a given *yield function* $\phi = \phi(\mathbf{P}, \mathbf{N}) : \text{SL} \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ whose sublevel $\{\phi(\mathbf{P}, \mathbf{N}) \leq 0\}$ represents the *elastic domain*. We assume that for all given $\mathbf{P} \in \text{SL}$ the yield function $\mathbf{N} \mapsto \phi(\mathbf{P}, \mathbf{N})$ is convex and that $\phi(\mathbf{P}, \mathbf{0}) < 0$.

Given the conjugacy of \mathbf{N} and \mathbf{P} from (2.5), we classically prescribe the flow rule in complementarity form as

$$\dot{\mathbf{P}} = \dot{z} \partial_{\mathbf{N}} \phi(\mathbf{P}, \mathbf{N}), \quad \dot{z} \geq 0, \quad \phi \leq 0, \quad \dot{z} \phi = 0. \quad (2.10)$$

This position falls within the class of *associated* plasticity models for the rate $\dot{\mathbf{P}}$ is prescribed to belong to the normal cone of the yield surface $\{\phi(\mathbf{P}, \mathbf{N}) = 0\}$. By dualization, this can be equivalently reformulated as

$$\mathbf{N} \in \partial_{\dot{\mathbf{P}}} R(\mathbf{P}, \dot{\mathbf{P}}) \quad (2.11)$$

where the *infinitesimal dissipation* $R(\mathbf{P}, \dot{\mathbf{P}})$ is the Legendre conjugate of the indicator function of the elastic domain $\{\phi(\mathbf{P}, \mathbf{N}) \leq 0\}$ with respect to its second argument.

2.7. Flow rule in terms of \mathbf{C}_p . We now aim at rewriting the flow rule (2.10) in terms of \mathbf{C}_p only. This follows again by assuming plastic-rotation invariance for the plastic-dissipation mechanism, namely

$$\phi(\mathbf{QP}, \mathbf{QN}) = \phi(\mathbf{P}, \mathbf{N}) \quad \forall \mathbf{Q} \in \text{SO} \quad (2.12)$$

and all $\mathbf{P} \in \text{SL}$ and $\mathbf{N} \in \mathbb{R}^{3 \times 3}$. Again, invariance with respect to rotations in \mathbf{P} corresponds to frame indifference in the intermediate configuration. On the other hand, under assumption (2.4) the tensor \mathbf{QN} is the thermodynamic force conjugated to \mathbf{QP} via (2.5).

By using (2.12), the decomposition $\mathbf{P} = \mathbf{RC}_p^{1/2}$, and relation (2.9) we define

$$\phi(\mathbf{P}, \mathbf{N}) = \phi(\mathbf{C}_p^{1/2}, \mathbf{C}_p^{1/2} \mathbf{T}) =: \hat{\phi}(\mathbf{C}_p, \mathbf{T})$$

and note that the function $\mathbf{T} \mapsto \hat{\phi}(\mathbf{C}_p, \mathbf{T})$ is convex and $\hat{\phi}(\mathbf{C}_p, \mathbf{0}) < 0$ for all given $\mathbf{C}_p \in \text{SL}_{\text{sym}}^+$. We aim now at showing that a flow rule in terms of $\dot{\mathbf{C}}_p$ follows from the flow rule (2.10). As \mathbf{T} is symmetric, we compute

$$\dot{\mathbf{P}} = \dot{z} \partial_{\mathbf{N}} \phi(\mathbf{P}, \mathbf{N}) \stackrel{(2.9)}{=} \dot{z} \partial_{\mathbf{N}} \hat{\phi}(\mathbf{C}_p, \mathbf{N}^\top \mathbf{P}^{-\top}) = \mathbf{P}^{-\top} \dot{z} \partial_{\mathbf{T}} \hat{\phi}(\mathbf{C}_p, \mathbf{T})$$

where we have also used that $\partial_{\mathbf{T}} \hat{\phi}(\mathbf{C}_p, \mathbf{T})$ is symmetric. Then, one has that

$$\dot{\mathbf{C}}_p = \dot{\mathbf{P}}^\top \mathbf{P} + \mathbf{P}^\top \dot{\mathbf{P}} = 2\dot{z} \partial_{\mathbf{T}} \hat{\phi}(\mathbf{C}_p, \mathbf{T}), \quad \dot{z} \geq 0, \quad \hat{\phi} \leq 0, \quad \dot{z} \hat{\phi} = 0. \quad (2.13)$$

This can be also expressed in the equivalent dual form

$$\mathbf{T} \in \partial_{\dot{\mathbf{C}}_p} \hat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p) \quad (2.14)$$

where the infinitesimal dissipation $\hat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p)$ is the Legendre conjugate of the indicator function of the elastic domain $\{\hat{\phi}(\mathbf{C}_p, \mathbf{T}) \leq 0\}$ with respect to \mathbf{T} . By using the plastic-rotation invariance (2.12) we deduce that

$$\begin{aligned} R(\mathbf{P}, \dot{\mathbf{P}}) &= \sup\{\mathbf{N} : \dot{\mathbf{P}} \mid \phi(\mathbf{P}, \mathbf{N}) \leq 0\} = \sup\{\mathbf{N} : \dot{\mathbf{P}} \mid \phi(\mathbf{QP}, \mathbf{QN}) \leq 0\} \\ &= \sup\{\mathbf{QN} : \mathbf{Q} \dot{\mathbf{P}} \mid \phi(\mathbf{QP}, \mathbf{QN}) \leq 0\} = R(\mathbf{QP}, \mathbf{Q} \dot{\mathbf{P}}) \quad \forall \mathbf{Q} \in \text{SO}. \end{aligned}$$

Correspondingly, we have that

$$\begin{aligned} \hat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p) &= \sup\{\mathbf{T} : \dot{\mathbf{C}}_p \mid \hat{\phi}(\mathbf{C}_p, \mathbf{T}) \leq 0\} = \sup\{\mathbf{T} : \dot{\mathbf{C}}_p \mid \phi(\mathbf{C}_p^{1/2}, \mathbf{C}_p^{1/2} \mathbf{T}) \leq 0\} \\ &= \sup\{\mathbf{C}_p^{1/2} \mathbf{T} : \mathbf{C}_p^{-1/2} \dot{\mathbf{C}}_p \mid \phi(\mathbf{C}_p^{1/2}, \mathbf{C}_p^{1/2} \mathbf{T}) \leq 0\} = R(\mathbf{C}_p^{1/2}, \mathbf{C}_p^{-1/2} \dot{\mathbf{C}}_p). \end{aligned} \quad (2.15)$$

Before closing this subsection, let us remark that the combination of frame (2.3) and plastic-rotation indifference (2.4), (2.12) entail that the model is invariant under the transformations $\mathbf{F}_e \rightarrow \mathbf{Q}\mathbf{F}_e\mathbf{R}$ and $\mathbf{P} \rightarrow \mathbf{R}\mathbf{P}$ with respect to all $\mathbf{Q}, \mathbf{R} \in \text{SO}$. This invariance is already advocated in [8, 28] as a natural requirement in relation with the multiplicative decomposition $\mathbf{F} = \mathbf{F}_e\mathbf{P}$, see also [53, Formula (4.5)].

2.8. Choice of the yield function. We shall now leave the abstract discussion of the previous subsections and choose the *yield function* as

$$\phi(\mathbf{P}, \mathbf{N}) := |\text{dev}(\mathbf{N}\mathbf{P}^\top)| - r.$$

Here $r > 0$ is a given *yield threshold* activating the plastic evolution. The latter choice of yield function is inspired by the classical von Mises theory and has to be traced back to Mandel [39], see also [30]. In particular, for all given $\mathbf{P} \in \text{SL}$, the function $\mathbf{N} \mapsto \phi(\mathbf{P}, \mathbf{N})$ is convex and $\phi(\mathbf{P}, \mathbf{0}) = -r < 0$. Moreover, ϕ fulfills plastic-rotation invariance (2.12). Correspondingly, the flow rule (2.10) is here specified as

$$\dot{\mathbf{P}}\mathbf{P}^{-1} \in \begin{cases} \dot{z} \frac{\text{dev}(\mathbf{N}\mathbf{P}^\top)}{|\text{dev}(\mathbf{N}\mathbf{P}^\top)|} & \text{for } \text{dev}(\mathbf{N}\mathbf{P}^\top) \neq 0, \\ \dot{z} \left\{ \mathbf{A} \in \mathbb{R}_{\text{dev}}^{3 \times 3} \mid |\mathbf{A}| \leq 1 \right\} & \text{for } \text{dev}(\mathbf{N}\mathbf{P}^\top) = 0. \end{cases} \quad (2.16)$$

The infinitesimal dissipation $R(\mathbf{P}, \dot{\mathbf{P}})$ reads

$$\begin{aligned} R(\mathbf{P}, \dot{\mathbf{P}}) &= \sup \{ \dot{\mathbf{P}} : \mathbf{N} \mid \phi(\mathbf{P}, \mathbf{N}) \leq 0 \} \\ &= \sup \{ \dot{\mathbf{P}}\mathbf{P}^{-1} : \mathbf{B} \mid |\text{dev} \mathbf{B}| \leq r \} = 2\tilde{R}(\dot{\mathbf{P}}\mathbf{P}^{-1}) \end{aligned}$$

with

$$\tilde{R}(\mathbf{A}) := \begin{cases} \frac{r}{2} |\mathbf{A}| & \text{if } \text{tr}(\mathbf{A}) = 0, \\ \infty & \text{else.} \end{cases} \quad (2.17)$$

Note that, owing to the definition of \mathbf{N} from (2.5), the flow rule in terms of \mathbf{P} reads

$$\partial_{\dot{\mathbf{P}}} R(\mathbf{P}, \dot{\mathbf{P}}) + \partial_{\mathbf{P}} W(\mathbf{C}, \mathbf{P}) \ni \mathbf{0}. \quad (2.18)$$

Let us now rewrite the flow rule in terms of \mathbf{C}_p . We have that

$$\hat{\phi}(\mathbf{C}_p, \mathbf{T}) = |\text{dev}(\mathbf{C}_p^{1/2} \mathbf{T} \mathbf{C}_p^{1/2})| - r$$

hence the flow rule (2.13) reads

$$\begin{aligned} \dot{\mathbf{C}}_p &\in 2 \begin{cases} \dot{\mathbf{z}} \mathbf{P}^\top \frac{\text{dev}(\mathbf{NP}^\top)}{|\text{dev}(\mathbf{NP}^\top)|} \mathbf{P} & \text{for } \text{dev}(\mathbf{NP}^\top) \neq 0, \\ \dot{\mathbf{z}} \left\{ \mathbf{A} \in \mathbb{R}_{\text{dev}}^{3 \times 3} \mid |\mathbf{A}| \leq 1 \right\} & \text{for } \text{dev}(\mathbf{NP}^\top) = 0, \end{cases} \\ &= 2 \begin{cases} \dot{\mathbf{z}} \mathbf{C}_p^{1/2} \frac{\text{dev}(\mathbf{C}_p^{1/2} \mathbf{T} \mathbf{C}_p^{1/2})}{|\text{dev}(\mathbf{C}_p^{1/2} \mathbf{T} \mathbf{C}_p^{1/2})|} \mathbf{C}_p^{1/2} & \text{for } \text{dev}(\mathbf{C}_p^{1/2} \mathbf{T} \mathbf{C}_p^{1/2}) \neq 0, \\ \dot{\mathbf{z}} \left\{ \mathbf{A} \in \mathbb{R}_{\text{dev}}^{3 \times 3} \mid |\mathbf{A}| \leq 1 \right\} & \text{for } \text{dev}(\mathbf{C}_p^{1/2} \mathbf{T} \mathbf{C}_p^{1/2}) = 0. \end{cases} \end{aligned} \quad (2.19)$$

Equivalently, by dualization we rewrite the flow rule in the form of (2.14) where the infinitesimal dissipation $\widehat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p)$ reads

$$\widehat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p) = R(\mathbf{C}_p^{1/2}, \mathbf{C}_p^{-1/2} \dot{\mathbf{C}}_p) = \widetilde{R}(\mathbf{C}_p^{-1/2} \dot{\mathbf{C}}_p \mathbf{C}_p^{-1/2})$$

in accordance with (2.15). In fact, given the specific form of \widetilde{R} we also have that

$$\widehat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p) = \widetilde{R}(\mathbf{C}_p^{-1} \dot{\mathbf{C}}_p) = \widetilde{R}(\dot{\mathbf{C}}_p \mathbf{C}_p^{-1}). \quad (2.20)$$

2.9. Material constitutive relation. By the definition of \mathbf{T} from (2.6), the flow rule (2.14) takes the compact form

$$\partial_{\dot{\mathbf{C}}_p} \widehat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p) + \partial_{\mathbf{C}_p} \widehat{W}(\mathbf{C}, \mathbf{C}_p) \ni \mathbf{0}. \quad (2.21)$$

Eventually, we have proved that the flow rule written in terms of $\dot{\mathbf{P}}$ (namely (2.10), (2.11), (2.16), or (2.18)) and the flow rule written in terms of $\dot{\mathbf{C}}_p$ (namely (2.13), (2.19), (2.14), or (2.21)) are such that solutions \mathbf{P} of the former correspond to solutions $\mathbf{C}_p = \mathbf{P}^\top \mathbf{P}$ of the latter. In the following we hence concentrate on the formulation (2.21).

Let us stress that the flow rule (2.19) induces an evolution in SL_{sym}^+ . First of all, as $\dot{\mathbf{C}}_p \mathbf{C}_p^{-1} = 2\dot{\mathbf{z}} \mathbf{C}_p^{1/2} \mathbf{D} \mathbf{C}_p^{-1/2}$ for some $\mathbf{D} \in \mathbb{R}_{\text{dev}}^{3 \times 3}$ with $|\mathbf{D}| \leq 1$, we have that

$$\text{tr}(\dot{\mathbf{C}}_p \mathbf{C}_p^{-1}) = 2\dot{\mathbf{z}} \text{tr}(\mathbf{C}_p^{1/2} \mathbf{D} \mathbf{C}_p^{-1/2}) = 2\dot{\mathbf{z}} \text{tr} \mathbf{D} = 0.$$

This implies by Jacobi's formula that

$$\frac{d}{dt} \det \mathbf{C}_p = \text{tr}(\dot{\mathbf{C}}_p \mathbf{C}_p^{-1}) = 0.$$

Hence, the evolution preserves the determinant constraint. This in particular entails that eigenvalues cannot change sign along smooth evolutions, so that positive definiteness is also conserved. Secondly, it is clear from the above expression (2.19) that $\dot{\mathbf{C}}_p$ is symmetric, so that evolution preserves symmetry as \mathbf{T} is symmetric. Note that the preservation of the determinant constraint follows solely from the choice of the flow rule. On the other hand, the symmetric character of the evolution is a combined effect of the form of the flow rule and of the energy.

As already commented in the Introduction, the possibility of reformulating the constitutive model in terms of \mathbf{C}_p instead of using \mathbf{P} is quite advantageous in terms of computational complexity. Indeed, \mathbf{C}_p belongs to the five-dimensional connected manifold SL_{sym}^+ whereas \mathbf{P} is in SL which is eight dimensional. Moreover, this brings also a computational advantage as matrix computations such as exponentials, logarithms, and powers are considerably faster on SL_{sym}^+ , see Appendix A. Finally, the fully Lagrangian formulation in \mathbf{C}_p requires no intermediate configurations. In particular, space discretizations can be based on the reference configuration only. The reader is referred to the recent [54] for a comparative discussion of the many finite-plasticity model based on \mathbf{C}_p available in the literature. The main result of [54] consists in proving in the isotropic case that all these constitutive relations coincide, and coincide to the one of this paper. Recall however that no isotropy in W_p is assumed throughout our analysis.

2.10. Dissipativity. Thanks to constitutive relation (2.7), we can express the dissipative character of the model by observing that

$$\frac{d}{dt}\widehat{W}(\mathbf{C}, \mathbf{C}_p) - \mathbf{S}:\frac{1}{2}\dot{\mathbf{C}} = -\mathbf{T}:\frac{1}{2}\dot{\mathbf{C}}_p \leq \widehat{R}(\mathbf{C}_p, \mathbf{0}) - \widehat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p) = -\widehat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p) \leq 0$$

where we have exploited the very definition of subdifferential and (2.14). In particular, for all sufficiently smooth evolutions we have that

$$\frac{d}{dt}\widehat{W}(\mathbf{C}, \mathbf{C}_p) \leq \mathbf{S}:\frac{1}{2}\dot{\mathbf{C}}.$$

2.11. Formulation via the logarithmic plastic strain. By using the isomorphism

$$\log : \text{SL}_{\text{sym}}^+ \rightarrow \mathbb{R}_{\text{dev}}^{3 \times 3},$$

the material constitutive model (2.21) can be equivalently reformulated in the variables

$$(\mathbf{C}, \log \mathbf{C}_p) \in \text{GL}_{\text{sym}}^+ \times \mathbb{R}_{\text{dev}}^{3 \times 3}.$$

An interesting feature of this choice is that the internal variable $\log \mathbf{C}_p$ takes values in the linear space $\mathbb{R}_{\text{dev}}^{3 \times 3}$.

3. ASIDE ON ENERGETIC SOLUTIONS

We collect here some notation and tools for the *energetic solvability* of general rate-independent systems. The notion of *energetic solutions* is by now classical [22, 42, 51] and we limit ourselves in collecting the minimal material needed along the analysis, by referring to the above-mentioned papers for all details, motivations, generalizations, and proofs.

Given a product of complete metric spaces $\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}$, an energy functional $\mathcal{E} : \mathcal{Q} \times [0, T] \rightarrow (-\infty, \infty]$, a dissipation distance $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \rightarrow [0, \infty]$ (see below for specific assumptions), and an initial datum $q_0 \in \mathcal{Q}$, we say that a trajectory $q = (y, z) : [0, T] \rightarrow \mathcal{Q}$

is an *energetic solution* corresponding to $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ starting from q_0 if $q(0) = q_0$ and, for all $t \in [0, T]$, the following two conditions are satisfied

$$q(t) \in \mathcal{S}(t) := \left\{ q \in \mathcal{Q} \mid \mathcal{E}(q, t) < \infty \text{ and } \mathcal{E}(q, t) \leq \mathcal{E}(\hat{q}, t) + \mathcal{D}(q, \hat{q}) \forall \hat{q} \in \mathcal{Q} \right\}, \quad (3.1)$$

$$\mathcal{E}(q(t), t) + \text{Diss}_{\mathcal{D}, [0, t]}(q) = \mathcal{E}(q(0), 0) + \int_0^t \partial_\tau \mathcal{E}(q(\tau), \tau) d\tau. \quad (3.2)$$

In the latter, we have denoted the *total dissipation* on $[0, t]$ by

$$\text{Diss}_{\mathcal{D}, [0, t]}(q) := \sup \left\{ \sum_{i=1}^N \mathcal{D}(q(t_{i-1}), q(t_i)) \right\}$$

where of course \mathcal{D} is intended to act just on the z component of q and the supremum is taken over all partitions $\{0 = t_0 \leq t_1 \leq \dots \leq t_N = t\}$ of $[0, t]$. Condition (3.1) is usually referred to as *global stability*. It expresses the optimality of the current state $q(t)$ against possible competitors \hat{q} with respect to the complementary energy, augmented by the dissipation from $q(t)$ to \hat{q} . Relation (3.2) imposes the balance between the actual complementary energy $\mathcal{E}(q(t), t)$ plus total dissipation $\text{Diss}_{\mathcal{D}, [0, t]}(q)$ and initial energy $\mathcal{E}(q(0), 0)$ plus work of the external actions $\int_0^t \partial_\tau \mathcal{E}(q(\tau), \tau) d\tau$. It hence corresponds to *energy conservation*.

We refer the reader to the above mentioned classical references and especially to the recent monograph [48] for a detailed discussion on the relevance of such a weak notion of solvability. Here we limit ourselves in observing that the energetic formulation is totally derivative-free, as no gradients of the functionals \mathcal{E} and \mathcal{D} nor of the trajectory q are involved. As such, it appears to be particularly well-suited for the nonsmooth case at hand.

A second important property of energetic solutions is that they naturally arise as limits of *incremental minimizations*. Assume to be given a partition $\{0 = t_0 < t_1 < \dots < t_N = T\}$ of the interval $[0, T]$. One is interested in incrementally solving the minimization problems

$$q_i = \text{Argmin} \{ \mathcal{E}(q, t_i) + \mathcal{D}(q_{i-1}, q) \mid q \in \mathcal{Q} \} \quad \text{for } i = 1, \dots, N. \quad (3.3)$$

These can be tackled by direct variational methods and, in particular, have at least a solution (q_0, q_1, \dots, q_N) under suitable coercivity and lower-semicontinuity assumptions for the *incremental* functional $q \mapsto \mathcal{E}(q, t_i) + \mathcal{D}(q_{i-1}, q)$. Then, one investigates the convergence of piecewise constant interpolants $\bar{q}^k(t)$ of sequences $(q_0, q_1^k, \dots, q_{N_k}^k)$ of solutions of (3.3) corresponding to partitions $\{0 = t_0^k < t_1^k < \dots < t_{N_k}^k = T\}$ with time step $\tau^k = \max(t_i^k - t_{i-1}^k)$ tending to zero. Under some specific qualification, see Lemma 3.1 below, one can prove that \bar{q}^k admits a convergent subsequence, whose limit is indeed an energetic solution corresponding to $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ [43]. We shall leave aside the discussion on the actual capability of energetic solutions of reproducing actual physical behaviors [47, 62, 67] and limit ourselves in recording that time-discretization is the most common tool for calculating approximate rate-independent evolutions. The study of energetic solutions bears a clear relevance for these are limits of time-discretizations.

In our analysis, we will use the following general existence result [42, 43].

Lemma 3.1 (Existence result). *Assume that*

$$\begin{aligned} &\mathcal{D} \text{ is lower semicontinuous and nondegenerate in } z, \text{ namely,} \\ &\mathcal{D}(q, \widehat{q}) \leq \mathcal{D}(q, \widetilde{q}) + \mathcal{D}(\widetilde{q}, \widehat{q}) \quad \forall q, \widehat{q}, \widetilde{q} \in \mathcal{Q} \quad \text{and} \quad \mathcal{D}(q, \widehat{q}) = 0 \Leftrightarrow z = \widehat{z}, \\ &\min\{\mathcal{D}(q_n, q), \mathcal{D}(q, q_n)\} \rightarrow 0 \Rightarrow z_n \rightarrow z, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\mathcal{E} \text{ has compact sublevels and controlled power } \partial_t \mathcal{E}, \text{ namely,} \\ &\{\mathcal{E} \leq \lambda\} \text{ is compact in } \mathcal{Q} \quad \forall t \in [0, T], \quad \forall \lambda \in \mathbb{R}, \\ &\exists c_1 > 0 \quad \forall (q_*, t_*) \text{ such that } \mathcal{E}(q_*, t_*) < \infty : \\ &\mathcal{E}(q_*, \cdot) \in C^1(0, T) \text{ and } |\partial_t \mathcal{E}(q_*, t)| \leq c_1(1 + \mathcal{E}(q_*, t)) \quad \forall t \in [0, T], \\ &\partial_t \mathcal{E} : \{\mathcal{E} \leq c_1\} \rightarrow \mathbb{R} \text{ is continuous,} \end{aligned} \tag{3.5}$$

Stable states are closed:

$$q_k \in \mathcal{S}(t_k) \text{ and } (q_k, t_k) \rightarrow (q_*, t_*) \Rightarrow q_* \in \mathcal{S}(t_*). \tag{3.6}$$

Then, for any $q_0 \in \mathcal{S}(0)$ and every partition $\{0 = t_0^k < t_1^k < \dots < t_{N^k}^k = T\}$ with time step $\tau^k = \max(t_i^k - t_{i-1}^k)$ the incremental minimization problems

$$q_i = \text{Argmin} \{ \mathcal{E}(q, t_i^k) + \mathcal{D}(q_{i-1}^k, q) \mid q \in \mathcal{Q} \} \quad \text{for } i = 1, \dots, N^k$$

admit a solution $\{q_0, q_1^k, \dots, q_{N^k}^k\}$. As $\tau^k \rightarrow 0$, the corresponding piecewise backward-constant interpolants $t \mapsto \bar{q}^k(t)$ on the partition admit a not relabeled subsequence such that, for all $t \in [0, T]$,

$$\bar{q}^k(t) \rightarrow q(t), \quad \text{Diss}_{\mathcal{D}, [0, t]}(\bar{q}^k) \rightarrow \text{Diss}_{\mathcal{D}, [0, t]}(q), \quad \mathcal{E}(\bar{q}^k(t), t) \rightarrow \mathcal{E}(q(t), t)$$

and $\partial_t \mathcal{E}(\bar{q}^k(\cdot), \cdot) \rightarrow \partial_t \mathcal{E}(q(\cdot), \cdot)$ in $L^1(0, T)$ where q is an energetic solution corresponding to $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ starting from $q(0)$.

In connection with the small-deformation case, we shall be confronted with the issue of the stability of energetic solutions with respect to limits. In this concern, we shall be using the following convergence tool from [49].

Lemma 3.2 (Evolutive Γ -convergence). *Assume to be given $\mathcal{D}_n : \mathcal{Q} \times \mathcal{Q} \rightarrow [0, \infty]$, $\mathcal{E}_n : \mathcal{Q} \times [0, T] \rightarrow (-\infty, \infty]$ for $n \in \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ such that \mathcal{D}_n and \mathcal{E}_n fulfill (3.4) and (3.5) uniformly with respect to $n \in \mathbb{N}_\infty$. Moreover, for all $n \in \mathbb{N}$ let $q_{0n} \in \mathcal{S}_n(0)$ be given, where $\mathcal{S}_n(t)$ indicates the set of stable states at time $t \in [0, 1]$ corresponding to*

$(\mathcal{Q}, \mathcal{E}_n, \mathcal{D}_n)$, and assume that

$$\mathcal{D}_\infty(q, \hat{q}) \leq \inf \left\{ \liminf_{n \rightarrow \infty} \mathcal{D}_n(q_n, \hat{q}_n) \mid \forall (q_n, \hat{q}_n) \rightarrow (q, \hat{q}) \right\} \quad \forall q, \hat{q} \in \mathcal{Q}, \quad (3.7)$$

$$\mathcal{E}_\infty(q, t) \leq \inf \left\{ \liminf_{n \rightarrow \infty} \mathcal{E}_n(q_n, t_n) \mid \forall (q_n, t_n) \rightarrow (q, t) \right\} \quad \forall (q, t) \in \mathcal{Q} \times [0, T], \quad (3.8)$$

$$\forall (q_n, t_n) \rightarrow (q, t) \text{ s. t. } q_n \in \mathcal{S}_n(t_n) \quad \forall \hat{q} \in \mathcal{Q} \quad \exists \hat{q}_n \in \mathcal{Q} :$$

$$\limsup_{n \rightarrow \infty} (\mathcal{E}_n(\hat{q}_n, t_n) - \mathcal{E}_n(q_n, t_n) + \mathcal{D}_n(\hat{q}_n, q_n)) \leq \mathcal{E}_\infty(\hat{q}, t) - \mathcal{E}_\infty(q, t) + \mathcal{D}_\infty(\hat{q}, q). \quad (3.9)$$

Let q_n be energetic solutions corresponding to $(\mathcal{Q}, \mathcal{E}_n, \mathcal{D}_n)$ starting from q_{0n} . If the initial values are well-prepared, namely

$$q_{0n} \rightarrow q_{0\infty} \quad \text{and} \quad \mathcal{E}(q_n(0), 0) \rightarrow \mathcal{E}_\infty(q_{0\infty}(0), 0),$$

there exists a not relabeled subsequence q_n such that $q_n(t) \rightarrow q_\infty(t)$ for all $t \in [0, T]$ where q_∞ is an energetic solution corresponding to $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$ starting from $q_{0\infty}$. Moreover, we have that, for all $t \in [0, T]$,

$$\text{Diss}_{\mathcal{D}_n, [0, t]}(q_n) \rightarrow \text{Diss}_{\mathcal{D}_\infty, [0, t]}(q_\infty), \quad \mathcal{E}_n(q_n(t), t) \rightarrow \mathcal{E}_\infty(q_\infty(t), t)$$

and $\partial_t \mathcal{E}_n(\bar{q}_n(\cdot), \cdot) \rightarrow \partial_t \mathcal{E}_\infty(q_\infty(\cdot), \cdot)$ in $L^1(0, T)$.

In particular, the stability of energetic solution in the limit $n \rightarrow \infty$ follows whenever one checks the *two separate* Γ -lim inf inequalities (3.7)-(3.8) and the *mutual recovery sequence* condition (3.9).

4. ENERGETIC SOLVABILITY OF THE CONSTITUTIVE MODEL

We devote this section to the discussion of the existence of energetic solutions to the constitutive model (2.21) at the material-point level. Assume to be given an initial state $\mathbf{C}_{p,0} \in \text{SL}_{\text{sym}}^+$ as well as the deformation history $t \in [0, T] \mapsto \mathbf{C}(t) \in \text{GL}_{\text{sym}}^+$. We are here interested in finding energetic solutions $t \in [0, T] \mapsto \mathbf{C}_p(t) \in \text{SL}_{\text{sym}}^+$ to the evolution problem (2.21), namely

$$\partial_{\dot{\mathbf{C}}_p} \widehat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p) + \partial_{\mathbf{C}_p} \widehat{W}(\mathbf{C}(t), \mathbf{C}_p) \ni \mathbf{0}, \quad \mathbf{C}_p(0) = \mathbf{C}_{p,0}. \quad (4.1)$$

To this aim, we set $q = z = \mathbf{C}_p$ (the elastic variable y plays no role here, indeed simplifying the argument) and indicate the complementary energy as $E(\mathbf{C}_p, t) := \widehat{W}(\mathbf{C}(t), \mathbf{C}_p)$.

We replace the infinitesimal dissipation R by the *dissipation metric* $D : \text{SL}_{\text{sym}}^+ \times \text{SL}_{\text{sym}}^+ \rightarrow [0, \infty]$ defined through the formula

$$D(\mathbf{C}_p, \widehat{\mathbf{C}}_p) := \inf \left\{ \int_0^1 \widehat{R}(\mathbf{C}_p(t), \dot{\mathbf{C}}_p(t)) dt \mid \mathbf{C}_p \in C^1(0, 1; \text{SL}_{\text{sym}}^+), \right. \\ \left. \mathbf{C}_p(0) = \mathbf{C}_p, \mathbf{C}_p(1) = \widehat{\mathbf{C}}_p \right\}. \quad (4.2)$$

As the function $\widehat{R}(\mathbf{C}_p, \cdot)$ is smooth for $\dot{\mathbf{C}}_p \neq 0$, positively 1-homogeneous, and has strictly convex square power, D results in a *Finsler metric* [46]. In particular, D fulfills the *triangle inequality*. Moreover, the actual choice of \widehat{R} entails that D is symmetric as well, see (2.20).

Before moving on, let us record some basic continuity and boundedness properties of the dissipation metric D in the following lemma.

Lemma 4.1. *The map D fulfills*

$$D(\mathbf{C}_p, \widehat{\mathbf{C}}_p) \leq \widetilde{R}(\log \mathbf{C}_p - \log \widehat{\mathbf{C}}_p) \quad \forall \mathbf{C}_p, \widehat{\mathbf{C}}_p \in \text{SL}_{\text{sym}}^+. \quad (4.3)$$

In particular, D is locally Lipschitz continuous and we have the bound

$$D(\mathbf{C}_p, \widehat{\mathbf{C}}_p) \leq 2r(|\mathbf{C}_p| + |\widehat{\mathbf{C}}_p| + 6) \quad \forall \mathbf{C}_p, \widehat{\mathbf{C}}_p \in \text{SL}_{\text{sym}}^+. \quad (4.4)$$

Proof. Given $\mathbf{C}_p \in \text{SL}_{\text{sym}}^+$, let $\mathbf{L} = \log \mathbf{C}_p \in \mathbb{R}_{\text{dev}}^{3 \times 3}$ and define the curve $t \in [0, 1] \mapsto \mathbf{C}(t) = \exp(t\mathbf{L}) \in \text{SL}_{\text{sym}}^+$ connecting \mathbf{I} and \mathbf{C}_p . Note that $\text{tr}(\mathbf{C}^{-1}\dot{\mathbf{C}}) = \text{tr}(\mathbf{L}) = 0$, so that $\widehat{R}(\mathbf{C}_p(t), \dot{\mathbf{C}}_p(t)) = r|\mathbf{L}|/2$, and

$$D(\mathbf{I}, \mathbf{C}_p) \leq \int_0^1 \widehat{R}(\mathbf{C}_p(t), \dot{\mathbf{C}}_p(t)) dt = \frac{r}{2}|\mathbf{L}|.$$

An analogous argument entails that $D(\mathbf{C}_p, \mathbf{I}) \leq r|\mathbf{L}|/2$. Let now $\lambda_3 \geq \lambda_2 \geq \lambda_1 > 0$ with $\lambda_1\lambda_2\lambda_3 = 1$ be the eigenvalues of \mathbf{C}_p . Then, $\mu_i = \log \lambda_i$ are the eigenvalues of \mathbf{L} . As we have that $\mu_1 + \mu_2 + \mu_3 = 0$, we deduce

$$|\mathbf{L}| \leq |\mu_1| + |\mu_2| + |\mu_3| \leq 4 \log \lambda_3 \leq 4(\lambda_3 - 1) \leq 4|\mathbf{C}_p - \mathbf{I}|.$$

Hence

$$D(\mathbf{I}, \mathbf{C}_p) \vee D(\mathbf{C}_p, \mathbf{I}) \leq 2r|\mathbf{C}_p - \mathbf{I}|.$$

By the triangle inequality, we hence obtain estimate (4.4).

Let now $\mathbf{C}_p, \widehat{\mathbf{C}}_p \in \text{SL}_{\text{sym}}^+$ be given and define $\mathbf{L} = \log \mathbf{C}_p$ and $\widehat{\mathbf{L}} = \log \widehat{\mathbf{C}}_p$. The curve $t \in [0, 1] \mapsto \mathbf{A}(t) := \exp(t\widehat{\mathbf{L}} + (1-t)\mathbf{L}) \in \text{SL}_{\text{sym}}^+$ connects \mathbf{C}_p and $\widehat{\mathbf{C}}_p$ and it is such that $\text{tr}(\mathbf{A}^{-1}\dot{\mathbf{A}}) = \text{tr}(\mathbf{L} - \widehat{\mathbf{L}}) = 0$. Hence

$$D(\mathbf{C}_p, \widehat{\mathbf{C}}_p) \leq \int_0^1 \widetilde{R}(\mathbf{A}^{-1}(t)\dot{\mathbf{A}}(t)) dt = \widetilde{R}(\mathbf{L} - \widehat{\mathbf{L}}) = \widetilde{R}(\log \mathbf{C}_p - \log \widehat{\mathbf{C}}_p)$$

so that the local Lipschitz continuity of D follows from that of the logarithm, see Appendix A. \square

We now focus on energetic solutions corresponding to $(\text{SL}_{\text{sym}}^+, E, D)$. In the following we will make use of the assumption

$$|\mathbf{F}_e^\top \partial_{\mathbf{F}_e} W_e(\mathbf{F}_e)| \leq c_2(1 + W_e(\mathbf{F}_e)) \quad \forall \mathbf{F}_e \in \text{GL}^+ \quad (4.5)$$

for some positive constant c_2 . Assumption (4.5) entails the controllability of the tensor $\mathbf{F}_e^\top \partial_{\mathbf{F}_e} W_e(\mathbf{F}_e)$ by means of the energy. It is a crucial condition in finite-deformation

theories [5, 6] and, in particular, is compatible with polyconvexity (see later on). Let us record that condition (4.5) has already been considered in the quasistatic context [22, 38, 50] and that it implies

$$|\partial_{\mathbf{F}_e} W_e(\mathbf{F}_e) \mathbf{F}_e^\top| \leq c(1 + W_e(\mathbf{F}_e)) \quad \forall \mathbf{F}_e \in \text{GL}^+ \quad (4.6)$$

for some c , depending on c_2 . This implication has been proved in [6, Prop. 2.3] for any frame-indifferent energy function $W_e(\mathbf{F}_e)$. With a completely similar argument, one can prove that, for isotropic functions $W_e(\mathbf{F}_e)$, (4.6) implies (4.5) so that these two conditions are equivalent in the frame of (2.4). We remark that (4.5)-(4.6) imply that W_e has polynomial growth [6, Prop. 2.7]. Let us anticipate that additional assumptions on W_e , in particular polyconvexity and coercivity, will be introduced later in Section 7. In addition to the control (4.5) we require \widehat{W}_p to be coercive. Namely, we ask that

$$\text{the sublevels of } \widehat{W}_p \text{ are compact.} \quad (4.7)$$

This coercivity condition will be progressively strengthened along the analysis, see (5.3), (7.2), and (8.3) later on.

Owing to the abstract existence result of Lemma 3.1 (here indeed simplified by the fact that $q = z = \mathbf{C}_p$) we have the following.

Theorem 4.2 (Energetic solvability of the constitutive material relation). *Assume (4.5) and (4.7). Let the deformation $t \mapsto \mathbf{C}(t) \in C^1(0, T)$ and the initial state $\mathbf{C}_{p,0} \in \mathcal{S}(0)$ be given, where $\mathcal{S}(t)$ denotes stable states at time $t \in [0, T]$ with respect to $(\text{SL}_{\text{sym}}^+, E, D)$. Then, there exists an energetic solution corresponding to $(\text{SL}_{\text{sym}}^+, E, D)$ starting from $\mathbf{C}_{p,0}$. More precisely, for all partitions $\{0 = t_0^k < t_1^k < \dots < t_{N^k}^k = T\}$ with time step $\tau^k = \max(t_i^k - t_{i-1}^k)$ the incremental minimization problems*

$$\mathbf{C}_{p,i} = \text{Argmin} \{E(\mathbf{C}_p, t_i^k) + D(\mathbf{C}_{p,i-1}, \mathbf{C}_{p,i}) \mid \mathbf{C}_p \in \text{SL}_{\text{sym}}^+\} \quad \text{for } i = 1, \dots, N^k$$

admit a solution $\{\mathbf{C}_{p,0}, \mathbf{C}_{p,1}^k, \dots, \mathbf{C}_{p,N^k}^k\}$ and, as $\tau^k \rightarrow 0$, the corresponding piecewise backward-constant interpolants $t \mapsto \overline{\mathbf{C}}_p^k(t)$ on the partition admit a not relabeled subsequence such that, for all $t \in [0, T]$,

$$\overline{\mathbf{C}}_p^k(t) \rightarrow \mathbf{C}_p(t), \quad \text{Diss}_{[0,t]}(\overline{\mathbf{C}}_p^k) \rightarrow \text{Diss}_{[0,t]}(\mathbf{C}_p), \quad E(\overline{\mathbf{C}}_p^k(t), t) \rightarrow E(\mathbf{C}_p(t), t),$$

and $\partial_t E(\overline{\mathbf{C}}_p^k(\cdot), \cdot) \rightarrow \partial_t E(\mathbf{C}_p(\cdot), \cdot)$ in $L^1(0, T)$ where \mathbf{C}_p is an energetic solution.

Proof. We limit ourselves in checking the assumptions of Lemma 3.1. Conditions (3.4) follow from Lemma 4.1. As $\widehat{W}_e \geq 0$ and \widehat{W}_p is coercive by (4.7), the compactness of the sublevels of $E(\cdot, t)$ ensues. The closure of the stable states (3.6) is a consequence of the continuity of E and D , see again Lemma 4.1.

We are left with the treatment of the power $\partial_t E(\mathbf{C}_p, t)$. Let us start by computing

$$\begin{aligned} \partial_t E(\mathbf{C}_p, t) &= \partial_t \widehat{W}_e(\mathbf{C}_e) = \partial_{\mathbf{C}_e} \widehat{W}(\mathbf{C}_e) : \dot{\mathbf{C}}_e \\ &= \partial_{\mathbf{C}_e} \widehat{W}(\mathbf{C}_p^{-1/2} \mathbf{C}(t) \mathbf{C}_p^{-1/2}) : \mathbf{C}_p^{-1/2} \dot{\mathbf{C}}(t) \mathbf{C}_p^{-1/2}. \end{aligned} \quad (4.8)$$

As $\widehat{W}_e \in C^1$ and the square root is continuous [29, pag. 23], the continuity of the map $\mathbf{C}_p \mapsto \partial_t E(\mathbf{C}_p, t)$ follows.

In order to prove the bound on the power in terms of the energy, recall that $\dot{\mathbf{C}}_e = \mathbf{P}^{-\top} \dot{\mathbf{C}} \mathbf{P}^{-1}$. Hence, we have that

$$\begin{aligned} \partial_t E(\mathbf{C}_p, t) &= \partial_{\mathbf{C}_e} \widehat{W}(\mathbf{C}_e) : \dot{\mathbf{C}}_e = \frac{1}{2} (\mathbf{F}_e^{-1} \partial_{\mathbf{F}_e} W(\mathbf{F}_e)) : (\mathbf{P}^{-\top} \dot{\mathbf{C}} \mathbf{P}^{-1}) \\ &= \frac{1}{2} (\partial_{\mathbf{F}_e} W(\mathbf{F}_e) \mathbf{F}_e^\top) : (\mathbf{F}_e^{-\top} \mathbf{P}^{-\top} \dot{\mathbf{C}} \mathbf{P}^{-1} \mathbf{F}_e^{-1}) \\ &= \frac{1}{2} (\partial_{\mathbf{F}_e} W(\mathbf{F}_e) \mathbf{F}_e^\top) : (\mathbf{F}^{-\top} \dot{\mathbf{C}} \mathbf{F}^{-1}). \end{aligned}$$

Note that the map $t \mapsto \mathbf{F}^{-\top}(t) \dot{\mathbf{C}}(t) \mathbf{F}^{-1}(t)$ is bounded as $t \mapsto \mathbf{C}(t) \in \text{GL}_{\text{sym}}^+$ is C^1 and $|\mathbf{F}^{-1}| = |\mathbf{C}^{-1/2}|$. By exploiting the control (4.6) we get

$$\begin{aligned} |\partial_t E(\mathbf{C}_p, t)| &\leq \frac{1}{2} c_2 (1 + W_e(\mathbf{F}_e)) |\mathbf{C}^{-1/2}(t) \dot{\mathbf{C}}(t) \mathbf{C}^{-1/2}(t)| \\ &\leq c (1 + \widehat{W}_e(\mathbf{C}_e)) \leq c (1 + E(\mathbf{C}_p, t)) \end{aligned} \tag{4.9}$$

which delivers the required bound. \square

5. SMALL-DEFORMATION LIMIT FOR THE CONSTITUTIVE MODEL

We turn now our attention to the study of the small-deformation case. The main result of this Section is a rigorous linearization limit for the constitutive model at the material-point level. This will follow from an application of the evolutive Γ -convergence Lemma 3.2. Linearization arguments are classically based on Taylor expansions for energy and dissipation densities. Here we concentrate instead on the proof of a variational convergence result. Indeed, we are here proving not only that the driving functionals are converging but, more significantly, that the whole trajectories converge. This brings to a rigorous *variational justification* of the linearization approach.

In order to tackle the small-deformation situation, we concentrate on suitably rescaled differences between \mathbf{C} or \mathbf{C}_p and the identity. In particular, given $\varepsilon > 0$ we reformulate the problem in the variables

$$\mathbf{e} := \frac{1}{2\varepsilon} (\mathbf{C} - \mathbf{I}) \in \mathbb{R}_{\text{sym}}^{3 \times 3}, \quad \mathbf{z} := \frac{1}{2\varepsilon} \log \mathbf{C}_p \in \mathbb{R}_{\text{dev}}^{3 \times 3}. \tag{5.1}$$

The tensor \mathbf{e} is nothing but the ε -rescaled *Green-Saint Venant* strain. By assuming $y = \text{id} + \varepsilon u$ where u is the rescaled displacement of the body, one has

$$\mathbf{C} = (\mathbf{I} + \varepsilon \nabla u)^\top (\mathbf{I} + \varepsilon \nabla u) = \mathbf{I} + 2\varepsilon \nabla u^{\text{sym}} + \varepsilon^2 \nabla u^\top \nabla u.$$

In particular $\nabla u^{\text{sym}} = (\nabla u + \nabla u^\top)/2$ corresponds to \mathbf{e} to first order.

The choice for \mathbf{z} is in the same spirit and corresponds to the ε -rescaled *Hencky logarithmic (plastic) strain*. Indeed $\mathbf{C}_p = \exp(2\varepsilon \mathbf{z})$ so that $\mathbf{C}_p \sim \mathbf{I} + 2\varepsilon \mathbf{z}$ to first order, in

analogy with the definition of $\mathbf{C} = \mathbf{I} + 2\varepsilon \mathbf{e}$. The different choice for \mathbf{z} is motivated by the nonlinear nature of the state space SL_{sym}^+ . In particular, we use here the fact that the logarithm is an isomorphism between SL_{sym}^+ and $\mathbb{R}_{\text{dev}}^{3 \times 3}$ in order to replace the nonlinear finite-plasticity state space SL_{sym}^+ with the linear space $\mathbb{R}_{\text{dev}}^{3 \times 3}$, corresponding indeed to the small-deformation limit. This is crucial in order to avoid the ε -dependence in the state spaces.

By using the equivalent variables (5.1) we introduce the rescaled energy density $W_\varepsilon : \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}_{\text{dev}}^{3 \times 3} \rightarrow [0, \infty]$ as

$$\begin{aligned} W_\varepsilon(\mathbf{e}, \mathbf{z}) &:= \frac{1}{\varepsilon^2} \widehat{W}(\mathbf{C}, \mathbf{C}_p) \\ &\stackrel{(5.1)}{=} \frac{1}{\varepsilon^2} \widehat{W}_e(\exp(-\varepsilon \mathbf{z})(\mathbf{I} + 2\varepsilon \mathbf{e}) \exp(-\varepsilon \mathbf{z})) + \frac{1}{\varepsilon^2} \widehat{W}_p(\exp(2\varepsilon \mathbf{z})). \end{aligned}$$

The relevance of this scaling is revealed for \widehat{W}_e and \widehat{W}_p twice differentiable at \mathbf{I} by computing Taylor expansions. In particular, by assuming with no loss of generality that the densities are normalized so that $\widehat{W}_e(\mathbf{I}) = \widehat{W}_p(\mathbf{I}) = 0$, that the reference configuration is stress-free ($\partial_{\mathbf{F}_e} W_e(\mathbf{I}) = \mathbf{0}$), and that the thermodynamic force \mathbf{T} conjugated to \mathbf{C}_p vanishes at non-plasticized states ($\partial_{\mathbf{C}_p} \widehat{W}_p(\mathbf{I}) = \mathbf{0}$), we compute

$$\begin{aligned} \widehat{W}_e(\mathbf{C}_p^{-1/2} \mathbf{C} \mathbf{C}_p^{-1/2}) &= \widehat{W}_e(\exp(-\varepsilon \mathbf{z})(\mathbf{I} + 2\varepsilon \mathbf{e}) \exp(-\varepsilon \mathbf{z})) \\ &= \frac{1}{2} \varepsilon^2 (\mathbf{e} - \mathbf{z}) : 4\partial_{\mathbf{C}_e}^2 \widehat{W}_e(\mathbf{I}) (\mathbf{e} - \mathbf{z}) + o(\varepsilon^2) = \frac{1}{2} \varepsilon^2 |\mathbf{e} - \mathbf{z}|_{\mathbb{C}}^2 + o(\varepsilon^2) \\ \widehat{W}_p(\mathbf{C}_p) &= \frac{1}{2} \varepsilon^2 \mathbf{z} : 4\partial_{\mathbf{C}_p}^2 \widehat{W}_p(\mathbf{I}) \mathbf{z} + o(\varepsilon^2) = \frac{1}{2} \varepsilon^2 |\mathbf{z}|_{\mathbb{H}}^2 + o(\varepsilon^2). \end{aligned}$$

We have here used the fact that $\exp(-\varepsilon \mathbf{z}) = \mathbf{I} - \varepsilon \mathbf{z} + o(\varepsilon)$ and defined the *elasticity* \mathbb{C} and *hardening* tensors \mathbb{H} as follows

$$\mathbb{C} := 4\partial_{\mathbf{C}_e}^2 \widehat{W}_e(\mathbf{I}) = \partial_{\mathbf{F}_e}^2 W_e(\mathbf{I}), \quad \mathbb{H} := 4\partial_{\mathbf{C}_p}^2 \widehat{W}_p(\mathbf{I}).$$

These fourth-order tensors are clearly symmetric, for they are Hessians. In addition, due to frame- and plastic-rotations indifference the tensors \mathbb{C} and \mathbb{H} present the so-called *minor symmetries* as well, namely

$$\mathbb{C}_{ijkl} = \mathbb{C}_{lkij} = \mathbb{C}_{ijkl}, \quad \mathbb{H}_{ijkl} = \mathbb{H}_{lkij} = \mathbb{H}_{ijkl}.$$

As for the dissipation metric, by rescaling D by 2ε we define $D_\varepsilon : \mathbb{R}_{\text{dev}}^{3 \times 3} \times \mathbb{R}_{\text{dev}}^{3 \times 3} \rightarrow [0, \infty]$ as

$$D_\varepsilon(\mathbf{z}_1, \mathbf{z}_2) := \frac{1}{2\varepsilon} D(\mathbf{C}_{p1}, \mathbf{C}_{p2}) \stackrel{(5.1)}{=} \frac{1}{2\varepsilon} D(\exp(2\varepsilon \mathbf{z}_1), \exp(2\varepsilon \mathbf{z}_2)).$$

Note that the scaling of the energy and of the dissipation is different for it corresponds for the different homogeneity of these terms.

Assume now to be given $t \in [0, T] \mapsto \mathbf{e}(t) \in \mathbb{R}_{\text{sym}}^{3 \times 3} \in C^1(0, T)$ and define accordingly the rescaled complementary energy densities $E_\varepsilon(\mathbf{z}, t) := W_\varepsilon(\mathbf{e}(t), \mathbf{z})$. Moreover, let the initial

values $\mathbf{z}_{0\varepsilon} \in \mathcal{S}_\varepsilon(0)$ be given, where $\mathcal{S}_\varepsilon(t)$ denotes the stable states at time $t \in [0, T]$ with respect to $(\mathbb{R}_{\text{dev}}^{3 \times 3}, E_\varepsilon, D_\varepsilon)$. By changing back variables via (5.1) one finds that $\mathbf{C}_{\text{p},0\varepsilon} = \exp(2\varepsilon \mathbf{z}_{0\varepsilon}) \in \mathcal{S}(0)$ where the latter denotes the stable states at $t = 0$ with respect to $(\text{SL}_{\text{sym}}^+, E/\varepsilon^2, D/(2\varepsilon))$. In particular, by virtue of Lemma 4.2 there exists an energetic solution $t \in [0, T] \mapsto \mathbf{z}_\varepsilon(t) \in \mathbb{R}_{\text{dev}}^{3 \times 3}$ corresponding to $(\mathbb{R}_{\text{dev}}^{3 \times 3}, E_\varepsilon, D_\varepsilon)$ and starting from $\mathbf{z}_{0\varepsilon}$. We shall term \mathbf{z}_ε a *finite-plasticity* trajectory in the following.

The focus of this section is to check that finite-plasticity trajectories \mathbf{z}_ε converge in the small-deformation limit $\varepsilon \rightarrow 0$ to the unique *linearized-plasticity* trajectory. The limiting linearized model is specified by letting

$$W_0(\mathbf{e}, \mathbf{z}) := \frac{1}{2}|\mathbf{e} - \mathbf{z}|_{\mathbb{C}}^2 + \frac{1}{2}|\mathbf{z}|_{\mathbb{H}}^2, \quad E_0(\mathbf{z}, t) := W_0(\mathbf{e}(t), \mathbf{z}), \quad D_0(\mathbf{z}, \widehat{\mathbf{z}}) := r|\widehat{\mathbf{z}} - \mathbf{z}|.$$

Given $\mathbf{z}_0 \in \mathbb{R}_{\text{dev}}^{3 \times 3}$, one can apply Lemma 3.1 and find an energetic solution corresponding to $(\mathbb{R}_{\text{dev}}^{3 \times 3}, E_0, D_0)$ and starting from \mathbf{z}_0 . As W_0 is quadratic, the latter energetic solution turns out to be a strong solution of the constitutive relation of linearized plasticity with linear kinematic hardening

$$r\partial|\mathbf{z}| + (\mathbb{C} + \mathbb{H})\mathbf{z} \in \mathbb{C}\mathbf{e}(t), \quad \mathbf{z}(0) = \mathbf{z}_0 \quad (5.2)$$

and it is thus unique [31]. We shall refer to this solution as the *linearized-plasticity trajectory* in the following.

The main result of this section reads as follows.

Theorem 5.1 (Small-deformation limit of the constitutive model). *Assume \widehat{W}_{p} to be coercive in the following sense*

$$\widehat{W}_{\text{p}}(\exp(2\mathbf{A})) \geq c_3|\mathbf{A}|^2 \quad \forall \mathbf{A} \in \mathbb{R}_{\text{dev}}^{3 \times 3} \quad (5.3)$$

where c_3 is a positive constant. Moreover, let \widehat{W}_{e} and \widehat{W}_{p} have quadratic behavior at identity, namely

$$\forall \delta > 0 \exists c_\delta > 0 \forall |\mathbf{A}| \leq c_\delta : \left| \widehat{W}_{\text{e}}(\mathbf{I} + 2\mathbf{A}) - \frac{1}{2}|\mathbf{A}|_{\mathbb{C}}^2 \right| + \left| \widehat{W}_{\text{p}}(\exp(2\mathbf{A})) - \frac{1}{2}|\mathbf{A}|_{\mathbb{H}}^2 \right| \leq \delta |\mathbf{A}|^2. \quad (5.4)$$

Let \mathbf{z}_ε be finite-plasticity trajectories starting from well-prepared initial data $\mathbf{z}_{0\varepsilon} \in \mathcal{S}_\varepsilon(0)$, namely

$$\mathbf{z}_{0\varepsilon} \rightarrow \mathbf{z}_0 \in \mathbb{R}_{\text{dev}}^{3 \times 3} \quad \text{and} \quad E_\varepsilon(\mathbf{z}_{0\varepsilon}, 0) \rightarrow E_0(\mathbf{z}_0, 0). \quad (5.5)$$

Then, for all $t \in [0, T]$

$$\mathbf{z}_\varepsilon(t) \rightarrow \mathbf{z}(t), \quad \text{Diss}_{D_\varepsilon, [0, t]}(\mathbf{z}_\varepsilon) \rightarrow \text{Diss}_{D_0, [0, t]}(\mathbf{z}), \quad E_\varepsilon(\mathbf{z}_\varepsilon(t), t) \rightarrow E_0(\mathbf{z}(t), t)$$

where \mathbf{z} is the unique linearized-plasticity trajectory starting from \mathbf{z}_0 .

Note that the coercivity condition (5.3) corresponds to a quantitative version of the weaker (4.7). Indeed, as \mathbf{A} is symmetric and deviatoric, large negative eigenvalues of \mathbf{A}

may arise only in presence of some large positive eigenvalue. In this case, the norm the exponential matrix is necessarily large as well.

Let us also remark that the quadratic behavior (5.4) of \widehat{W}_e is equivalent to the following

$$\forall \delta > 0 \exists \tilde{c}_\delta > 0 \forall |\mathbf{A}| \leq \tilde{c}_\delta : \quad \left| W_e(\mathbf{I} + \mathbf{A}) - \frac{1}{2} |\mathbf{A}|_{\mathbb{C}}^2 \right| \leq \delta |\mathbf{A}|^2. \quad (5.6)$$

Condition (5.4) implies in particular that \widehat{W}_e and \widehat{W}_p are twice differentiable at the identity and

$$\begin{aligned} \widehat{W}_e(\mathbf{I}) = W_p(\mathbf{I}) = 0, \quad \partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{I}) = \partial_{\mathbf{C}_p} \widehat{W}_p(\mathbf{I}) = \mathbf{0}, \\ 4\partial_{\mathbf{C}_e}^2 \widehat{W}_e(\mathbf{I}) = \mathbb{C}, \quad 4\partial_{\mathbf{C}_p}^2 \widehat{W}_p(\mathbf{I}) = \mathbb{H}. \end{aligned}$$

On the other hand, these conditions imply (5.4) in case \widehat{W}_e and \widehat{W}_p are C^2 in a neighborhood of the identity.

Let us start by preparing some convergence lemmas for the energy density and the dissipation metrics.

Lemma 5.2 (Convergence of E_ε). *Under the assumptions of Theorem 5.1 we have that $E_\varepsilon \rightarrow E_0$ locally uniformly in \mathbf{z} and uniformly in t .*

Proof. Let $\mathbf{z} \in \text{SL}_{\text{sym}}^+$. We have that $\exp(-\varepsilon \mathbf{z}) = \mathbf{I} - \varepsilon \mathbf{z} + \varepsilon^2 \mathbf{L}$, where \mathbf{L} is bounded in terms of $|\mathbf{z}|$ only. In particular, we have that

$$\begin{aligned} \exp(-\varepsilon \mathbf{z})(\mathbf{I} + 2\varepsilon \mathbf{e}(t)) \exp(-\varepsilon \mathbf{z}) &= (\mathbf{I} - \varepsilon \mathbf{z} + \varepsilon^2 \mathbf{L})(\mathbf{I} + 2\varepsilon \mathbf{e}(t))(\mathbf{I} - \varepsilon \mathbf{z} + \varepsilon^2 \mathbf{L}) \\ &= \mathbf{I} + \varepsilon(\mathbf{e}(t) - \mathbf{z}) + \varepsilon^2 \widehat{\mathbf{L}} \end{aligned}$$

where the matrix $\widehat{\mathbf{L}}$ is bounded in terms of $\|\mathbf{e}\|_{L^\infty}$ and $|\mathbf{z}|$ only. Let now $\delta > 0$ and $c_\delta > 0$ from (5.4) be given and let ε so small that $|\varepsilon(\mathbf{e}(t) - \mathbf{z}) + \varepsilon^2 \widehat{\mathbf{L}}| + |\varepsilon \mathbf{z}| \leq c_\delta$. Such an ε depends on $\|\mathbf{e}\|_{L^\infty}$ and $|\mathbf{z}|$. Then, by (5.4) we have that

$$\begin{aligned} |E_\varepsilon(\mathbf{z}, t) - E_0(\mathbf{z}, t)| &= \left| \frac{1}{\varepsilon^2} \widehat{W}_e(\mathbf{I} + \varepsilon(\mathbf{e}(t) - \mathbf{z}) + \varepsilon^2 \widehat{\mathbf{L}}) + \frac{1}{\varepsilon^2} \widehat{W}_p(\exp(2\varepsilon \mathbf{z})) - \frac{1}{2} |\mathbf{e}(t) - \mathbf{z}|_{\mathbb{C}}^2 - \frac{1}{2} |\mathbf{z}|_{\mathbb{H}}^2 \right| \\ &\leq \left| \frac{1}{2} |\mathbf{e}(t) - \mathbf{z} + \varepsilon \widehat{\mathbf{L}}|_{\mathbb{C}}^2 - \frac{1}{2} |\mathbf{e}(t) - \mathbf{z}|_{\mathbb{C}}^2 \right| + \delta |(\mathbf{e}(t) - \mathbf{z}) + \varepsilon \widehat{\mathbf{L}}|^2 + \delta |\mathbf{z}|^2 \leq c(\varepsilon + \delta) \end{aligned}$$

where the positive constant c depends on $\|\mathbf{e}\|_{L^\infty}$ and $|\mathbf{z}|$. As $\delta > 0$ is arbitrary the local uniform convergence follows. \square

Lemma 5.3 (Convergence of D_ε). *Under the assumptions of Theorem 5.1 we have that $D_\varepsilon \rightarrow D_0$ pointwise and*

$$\widehat{R}(\widehat{\mathbf{z}} - \mathbf{z}) = D_0(\mathbf{z}, \widehat{\mathbf{z}}) \leq \liminf_{\varepsilon \rightarrow 0} D_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \quad \forall (\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \rightarrow (\mathbf{z}, \widehat{\mathbf{z}}), \quad (5.7)$$

In particular $D_\varepsilon \rightarrow D_0$ in the sense of Γ -convergence [12, 17].

Proof. Let us start by proving the Γ -lim inf inequality (5.7). Assume to be given $(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \rightarrow (\mathbf{z}, \widehat{\mathbf{z}})$ so that, with no loss of generality, $\sup_\varepsilon D_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) < \infty$. For all ε small, let $\mathbf{C}_\varepsilon \in C^1(0, 1; \text{SL}_{\text{sym}}^+)$ be such that

$$(1-\varepsilon) \int_0^1 \widetilde{R}(\dot{\mathbf{C}}_\varepsilon \mathbf{C}_\varepsilon^{-1}) dt \leq D(\exp(2\varepsilon \mathbf{z}_\varepsilon), \exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon))$$

along with $\mathbf{C}_\varepsilon(0) = \exp(2\varepsilon \mathbf{z}_\varepsilon)$ and $\mathbf{C}_\varepsilon(1) = \exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon)$. In particular, $\dot{\mathbf{C}}_\varepsilon \mathbf{C}_\varepsilon^{-1} \in \mathbb{R}_{\text{dev}}^{3 \times 3}$ almost everywhere. By possibly reparametrizing time, we can additionally assume that

$$\widetilde{R}(\dot{\mathbf{C}}_\varepsilon \mathbf{C}_\varepsilon^{-1}) \leq 2D(\exp(2\varepsilon \mathbf{z}_\varepsilon), \exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon)).$$

Let us now estimate the distance of \mathbf{C}_ε from the identity as follows

$$\begin{aligned} |\mathbf{C}_\varepsilon(t) - \mathbf{I}| &\leq \int_0^1 |\dot{\mathbf{C}}_\varepsilon \mathbf{C}_\varepsilon^{-1}| |\mathbf{C}_\varepsilon| dt + |\mathbf{C}_\varepsilon(0) - \mathbf{I}| \\ &\stackrel{(2.17)}{=} \frac{2}{r} \int_0^1 \widetilde{R}(\dot{\mathbf{C}}_\varepsilon \mathbf{C}_\varepsilon^{-1}) |\mathbf{C}_\varepsilon| dt + |\exp(2\varepsilon \mathbf{z}_{0\varepsilon}) - \mathbf{I}| \\ &\leq \frac{2}{r} \sup_{t \in [0,1]} |\mathbf{C}_\varepsilon(t)| \int_0^1 \widetilde{R}(\dot{\mathbf{C}}_\varepsilon \mathbf{C}_\varepsilon^{-1}) dt + c\varepsilon \leq \frac{2}{r} \sup_{t \in [0,1]} |\mathbf{C}_\varepsilon(t)| 2D(\exp(2\varepsilon \mathbf{z}_\varepsilon), \exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon)) + c\varepsilon \\ &\stackrel{(4.3)}{\leq} \frac{2}{r} \sup_{t \in [0,1]} |\mathbf{C}_\varepsilon(t)| 4\varepsilon \widetilde{R}(\widehat{\mathbf{z}}_\varepsilon - \mathbf{z}_\varepsilon) + c\varepsilon = 4\varepsilon \sup_{t \in [0,1]} |\mathbf{C}_\varepsilon(t)| |\widehat{\mathbf{z}}_\varepsilon - \mathbf{z}_\varepsilon| + c\varepsilon \\ &\leq 4\varepsilon \left(3 + \sup_{t \in [0,1]} |\mathbf{C}_\varepsilon(t) - \mathbf{I}|\right) |\widehat{\mathbf{z}}_\varepsilon - \mathbf{z}_\varepsilon| + c\varepsilon. \end{aligned}$$

Hence, $\mathbf{C}_\varepsilon \rightarrow \mathbf{I}$ uniformly. Clearly $\mathbf{C}_\varepsilon^{-1} = \text{cof } \mathbf{C}_\varepsilon^\top \rightarrow \text{cof } \mathbf{I}^\top = \mathbf{I}$ uniformly as well. Let us now define $\widehat{\mathbf{C}}_\varepsilon = \mathbf{I} + (\mathbf{C}_\varepsilon - \mathbf{I})/(2\varepsilon)$ and use $\widehat{\mathbf{C}}_\varepsilon = \dot{\mathbf{C}}_\varepsilon/(2\varepsilon)$ in order to compute that

$$\begin{aligned} |\dot{\widehat{\mathbf{C}}}_\varepsilon| &\leq |\dot{\mathbf{C}}_\varepsilon \mathbf{C}_\varepsilon^{-1}| |\mathbf{C}_\varepsilon| \leq c \widetilde{R}(\dot{\mathbf{C}}_\varepsilon \mathbf{C}_\varepsilon^{-1}) = \frac{c}{2\varepsilon} \widetilde{R}(\dot{\mathbf{C}}_\varepsilon \mathbf{C}_\varepsilon^{-1}) \\ &\leq \frac{c}{2\varepsilon} 2D(\exp(2\varepsilon \mathbf{z}_\varepsilon), \exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon)) = cD_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \leq c. \end{aligned}$$

Up to a not relabeled subsequence we have that $\widehat{\mathbf{C}}_\varepsilon \rightarrow \widehat{\mathbf{C}}$ weakly star in $W^{1,\infty}(0, 1; \text{SL}_{\text{sym}}^+)$. By making use of the lower-semicontinuity Lemma C.1 we conclude that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} D_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) &= \liminf_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} D(\exp(2\varepsilon \mathbf{z}_\varepsilon), \exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon)) \geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^1 \widetilde{R}(\dot{\mathbf{C}}_\varepsilon \mathbf{C}_\varepsilon^{-1}) dt \\ &= \liminf_{\varepsilon \rightarrow 0} \int_0^1 \widetilde{R}(\dot{\widehat{\mathbf{C}}}_\varepsilon \mathbf{C}_\varepsilon^{-1}) dt \geq \int_0^1 \widetilde{R}(\dot{\widehat{\mathbf{C}}}) dt \geq \widetilde{R}(\widehat{\mathbf{C}}(1) - \widehat{\mathbf{C}}(0)). \end{aligned}$$

Relation (5.7) follows by noting that

$$\widehat{\mathbf{C}}(1) - \widehat{\mathbf{C}}(0) = \lim_{\varepsilon \rightarrow 0} (\widehat{\mathbf{C}}_\varepsilon(1) - \widehat{\mathbf{C}}_\varepsilon(0)) = \lim_{\varepsilon \rightarrow 0} \frac{\exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon) - \exp(2\varepsilon \mathbf{z}_\varepsilon)}{2\varepsilon} = \widehat{\mathbf{z}} - \mathbf{z}.$$

We now turn to the proof of the pointwise convergence. Let $(\mathbf{z}, \widehat{\mathbf{z}}) \in \text{SL}_{\text{sym}}^+ \times \text{SL}_{\text{sym}}^+$ be given and recall that

$$D_\varepsilon(\mathbf{z}, \widehat{\mathbf{z}}) = \frac{1}{2\varepsilon} D(\exp(2\varepsilon\mathbf{z}), \exp(2\varepsilon\widehat{\mathbf{z}})) \stackrel{(4.3)}{\leq} \widetilde{R}(\widehat{\mathbf{z}} - \mathbf{z}).$$

By using also (5.7) we conclude that $D_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \rightarrow \widetilde{R}(\widehat{\mathbf{z}} - \mathbf{z}) = D_0(\widehat{\mathbf{z}} - \mathbf{z})$. \square

Proof of Theorem 5.1. We aim at applying the evolutive Γ -convergence Lemma 3.2 to the sequence of functionals $(E_\varepsilon, D_\varepsilon)$ and the corresponding limit (E_0, D_0) . Note that these are defined on the common state space $\mathbb{R}_{\text{dev}}^{3 \times 3}$. Conditions (3.4) readily follow for all D_ε and D_0 . The uniform compactness of the sublevels of E_ε is a consequence of the coercivity (5.3) whereas the smoothness and the uniform control on the powers can be obtained by arguing along the lines of the proof of Theorem 4.2, in particular as in (4.9).

The Γ -lim inf properties (3.7)-(3.8) follow from Lemmas 5.2-5.3. As for the closure condition (3.9) assume to be given $\mathbf{z}_\varepsilon \in \mathcal{S}(t_\varepsilon)$ so that $(\mathbf{z}_\varepsilon, t_\varepsilon) \rightarrow (\mathbf{z}, t)$ and $\widehat{\mathbf{z}} \in \mathbb{R}_{\text{dev}}^{3 \times 3}$. Then, by choosing the constant (mutual recovery) sequence $\widehat{\mathbf{z}}_\varepsilon = \widehat{\mathbf{z}}$ we readily compute that

$$\begin{aligned} E_\varepsilon(\widehat{\mathbf{z}}_\varepsilon, t_\varepsilon) - E_\varepsilon(\mathbf{z}_\varepsilon, t_\varepsilon) + D_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \\ = E_\varepsilon(\widehat{\mathbf{z}}_\varepsilon, t_\varepsilon) - E_\varepsilon(\mathbf{z}_\varepsilon, t_\varepsilon) + \frac{1}{2\varepsilon} D(\exp(2\varepsilon\mathbf{z}_\varepsilon), \exp(2\varepsilon\widehat{\mathbf{z}})) \\ \stackrel{(4.3)}{\leq} E_\varepsilon(\widehat{\mathbf{z}}, t_\varepsilon) - E_\varepsilon(\mathbf{z}_\varepsilon, t_\varepsilon) + \widetilde{R}(\widehat{\mathbf{z}} - \mathbf{z}_\varepsilon). \end{aligned}$$

In particular, by exploiting the local uniform convergence $E_\varepsilon \rightarrow E_0$ from Lemma 5.2, the smoothness of W_e and \mathbf{e} , and the continuity of \widetilde{R} , we conclude for (3.9). \square

Before closing this section, let us record here the convergence of the energy densities, which will turn out useful in Section 8.

Lemma 5.4 (Energy densities convergence). *Under the assumptions (5.4) we have the continuous convergences*

$$\mathbf{A}_\varepsilon \rightarrow \mathbf{A} \Rightarrow \frac{1}{\varepsilon^2} W_e(\mathbf{I} + \varepsilon \mathbf{A}_\varepsilon) \rightarrow \frac{1}{2} |\mathbf{A}|_{\mathbb{C}}^2, \quad (5.8)$$

$$\mathbf{B}_\varepsilon \rightarrow \mathbf{B} \Rightarrow \frac{1}{\varepsilon^2} \widehat{W}_p(\exp(2\varepsilon \mathbf{B}_\varepsilon)) \rightarrow \frac{1}{2} |\mathbf{B}|_{\mathbb{H}}^2. \quad (5.9)$$

Indeed, the latter convergences are locally uniform.

Proof. Let $\mathbf{B}_\varepsilon \rightarrow \mathbf{B}$, define $M = \sup |\mathbf{B}_\varepsilon|$, and fix $\delta > 0$. Due to the quadratic behavior (5.4) of \widehat{W}_p , for all $\varepsilon < \delta/M$ we have

$$\left| \frac{1}{\varepsilon^2} \widehat{W}_p(\exp(2\varepsilon \mathbf{B}_\varepsilon)) - \frac{1}{2} |\mathbf{B}_\varepsilon|^2 \right| \leq \delta M^2$$

and relation (5.9) follows by passing to the lim sup in ε as δ is arbitrary. The proof of (5.8) is analogous and follows by considering the equivalent (5.6). \square

6. QUASISTATIC EVOLUTION

We turn now to the formulation of the quasistatic evolution problem. In particular, we address the coupling of the constitutive model at the material point and the boundary-value problem for the quasistatic elastic response.

6.1. Quasistatic equilibrium system. Let $\Gamma_{\text{tr}}, \Gamma_{\text{D}} \subset \partial\Omega$ be measurable and relatively open in $\partial\Omega$ so that $\Gamma_{\text{D}} \subset \partial\Omega$ has positive surface measure, $\Gamma_{\text{tr}} \cap \Gamma_{\text{D}} = \emptyset$, and $\Gamma_{\text{tr}} \cup \Gamma_{\text{D}} = \partial\Omega$. The set Γ_{tr} represents the portion of the boundary where traction is exerted. On the other hand, Γ_{D} is the boundary part subject to no deformation.

We neglect inertial effects and concentrate on a quasistatic approximation of the time evolution. The evolution is hence expressed via the equilibrium system

$$\nabla \cdot \boldsymbol{\sigma} + b(t) = 0 \quad \text{in } \Omega \times (0, T). \quad (6.1)$$

along with the position of the *total stress*

$$\boldsymbol{\sigma} := \partial_{\nabla y} W_e(\nabla y \mathbf{C}_p^{-1/2}) = \partial_{\mathbf{F}_e} W_e(\nabla y \mathbf{C}_p^{-1/2}) \mathbf{C}_p^{-1/2},$$

and of the boundary conditions

$$y = \text{id} \quad \text{in } \Gamma_{\text{D}} \times (0, T), \quad (6.2)$$

$$\boldsymbol{\sigma} \nu = \tau(t) \quad \text{in } \Gamma_{\text{tr}} \times (0, T) \quad (6.3)$$

where $b : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ and $\tau : \Gamma_{\text{tr}} \times (0, T) \rightarrow \mathbb{R}^3$ are given body force and traction, respectively, and ν indicates the outward normal to $\partial\Omega$. Let us comment that our choice of boundary conditions is dictated by simplicity. Nonconstant imposed boundary deformations could be considered as well by following [22].

6.2. Energy. We specify the *total energy* of the medium as

$$\mathcal{E}(y, \mathbf{C}_p, t) = \mathcal{W}(y, \mathbf{C}_p) - \langle \ell(t), y \rangle \quad (6.4)$$

namely as the sum of the *stored energy* $\mathcal{W}(y, \mathbf{C}_p)$ and the work of *external action* $\langle \ell(t), y \rangle$.

The stored energy $\mathcal{W}(y, \mathbf{C}_p)$ is defined as

$$\mathcal{W}(y, \mathbf{C}_p) = \int_{\Omega} \left(W_e(\nabla y \mathbf{C}_p^{-1/2}) + \widehat{W}_p(\mathbf{C}_p) + \frac{\mu}{r} |\nabla \mathbf{C}_p|^r \right) dx \quad (6.5)$$

and results from the sum of the *elastic*, the *plastic*, and a *gradient* energy term. The first two terms have already been introduced above.

As anticipated in the Introduction, at the quasistatic evolution level the presence of the gradient term in $\nabla \mathbf{C}_p$ with $\mu > 0$ and $r \geq 1$ sets our problem within the class of gradient plasticity models [23, 24, 52]. A gradient term $\nabla \mathbf{P}$ is considered in the quasistatic evolution analysis in [38] as well. Although the compactifying effects of the two terms $\nabla \mathbf{C}_p$

and $\nabla \mathbf{P}$ are comparable, note that such terms deliver different contributions. Indeed, one can compute

$$|\nabla \mathbf{C}_p| = |\nabla(\mathbf{P}^\top \mathbf{P})| = |(\mathbf{P}^\top \nabla \mathbf{P}^\top)^\top + \mathbf{P}^\top \nabla \mathbf{P}| \leq |\mathbf{P}^\top \nabla \mathbf{P}^\top| + |\mathbf{P}^\top \nabla \mathbf{P}| \leq 2|\mathbf{P}| |\nabla \mathbf{P}|.$$

In particular, the term $|\nabla \mathbf{P}|$ controls $|\nabla \mathbf{C}_p|$ for bounded \mathbf{P} . On the other hand, the term $|\nabla \mathbf{C}_p|$ vanishes on SO, hence cannot control $|\nabla \mathbf{P}|$.

The evolution is steered by external actions. In particular, the external work reads

$$\langle \ell(t), y \rangle := \int_{\Omega} b(x, t) \cdot y(x) \, dx + \int_{\Gamma_{tr}} \tau(x, t) \cdot y(x) \, dS$$

where dS denotes the surface measure on $\partial\Omega$.

6.3. Flow rule. Let us introduce a notation for the *total energy density* in (6.5) as

$$\widehat{W}(\mathbf{C}, \mathbf{C}_p, \nabla \mathbf{C}_p) = \widehat{W}_e(\mathbf{C}_p^{-1/2} \mathbf{C} \mathbf{C}_p^{-1/2}) + \widehat{W}_p(\mathbf{C}_p) + \frac{\mu}{r} |\nabla \mathbf{C}_p|^r.$$

Then, the *second Piola-Kirchhoff stress* tensor \mathbf{S} is again defined as

$$\mathbf{S} := 2\partial_{\mathbf{C}} \widehat{W}(\mathbf{C}, \mathbf{C}_p, \nabla \mathbf{C}_p) = \mathbf{C}_p^{-1/2} : 2\partial_{\mathbf{C}_e} \widehat{W}_e(\mathbf{C}_e) : \mathbf{C}_p^{-1/2} \in \mathbb{R}_{\text{sym}}^{3 \times 3}.$$

Let again \mathbf{T} denote the thermodynamic force conjugated to \mathbf{C}_p . Because of the gradient term $\nabla \mathbf{C}_p$, the partial derivative is replaced by the functional variation and \mathbf{T} can be split into a *local* and *nonlocal* part as follows

$$\mathbf{T} := -\delta_{\mathbf{C}_p} \widehat{W} = -\partial_{\mathbf{C}_p} \widehat{W} + \nabla \cdot \partial_{\nabla \mathbf{C}_p} \widehat{W} \in \mathbb{R}_{\text{sym}}^{3 \times 3}.$$

Here, the symbol $\delta_{\mathbf{C}_p}$ refers to some suitable functional variation. The flow rule (2.21) takes here the form

$$\partial_{\dot{\mathbf{C}}_p} \widehat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p) + \delta_{\mathbf{C}_p} \widehat{W}(y, \mathbf{C}_p) \ni \mathbf{0}$$

where

$$\delta_{\mathbf{C}_p} \widehat{W} = \partial_{\mathbf{C}_p} W_e(\nabla y \mathbf{C}_p^{-1/2}) + \partial_{\mathbf{C}_p} \widehat{W}_h(\mathbf{C}_p) - \mu \nabla \cdot (|\nabla \mathbf{C}_p|^{r-2} \nabla \mathbf{C}_p).$$

Owing to the above introduced notation, we can formally summarize the relations governing the quasistatic evolution problem as

$$\nabla \cdot (\partial_{\mathbf{F}_e} W_e(\nabla y \mathbf{C}_p^{-1/2}) \mathbf{C}_p^{-1/2}) + b(t) = \mathbf{0}, \quad (6.6)$$

$$\partial_{\dot{\mathbf{C}}_p} \widehat{R}(\mathbf{C}_p, \dot{\mathbf{C}}_p) + \partial_{\mathbf{C}_p} W_e(\nabla y \mathbf{C}_p^{-1/2}) + \partial_{\mathbf{C}_p} \widehat{W}_p(\mathbf{C}_p) - \mu \nabla \cdot (|\nabla \mathbf{C}_p|^{r-2} \nabla \mathbf{C}_p) \ni 0, \quad (6.7)$$

to be complemented with boundary and initial conditions. This strong formulation (6.6)-(6.7) of the quasistatic evolution problem seems at present inaccessible from the point of view of the existence of solutions. We are hence forced in resorting to a weak-solution concept.

7. ENERGETIC SOLVABILITY OF THE QUASISTATIC-EVOLUTION PROBLEM

The aim of this section is to present the energetic formulation of system (6.6)-(6.7). We start by introducing suitable functional spaces for the state variables. Let the coefficients $q_y, q_p, r > 1$ be given, specific assumptions are introduced below. We will ask that the deformation y belongs to

$$\mathcal{Y} := \{y \in W^{1,q_y}(\Omega, \mathbb{R}^3) \mid y = \text{id on } \Gamma_D\},$$

The state space for \mathbf{C}_p is then defined by

$$\mathcal{Z} = \{\mathbf{C}_p \in L^{q_p}(\Omega, \mathbb{R}^{3 \times 3}) \cap W^{1,r}(\Omega, \mathbb{R}^{3 \times 3}) \mid \mathbf{C}_p(x) \in \text{SL}_{\text{sym}}^+ \text{ for a.e. } x \in \Omega\}.$$

\mathcal{Y} and \mathcal{Z} are weakly closed subsets of separable and reflexive Banach spaces. Finally, we set $\mathcal{Q} := \mathcal{Y} \times \mathcal{Z}$.

The total energy $\mathcal{E} : \mathcal{Y} \times \mathcal{Z} \times [0, T] \rightarrow (-\infty, \infty]$ has been introduced in (6.4). As for the dissipation metric we let $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \rightarrow [0, \infty]$ be defined by

$$\mathcal{D}(\mathbf{C}_p, \widehat{\mathbf{C}}_p) := \int_{\Omega} D(\mathbf{C}_p(x), \widehat{\mathbf{C}}_p(x)) \, dx$$

where D is the dissipation metric introduced in (4.2). The local Lipschitz continuity of D , see Lemma 4.1, translates to an analogous statement for \mathcal{D} .

The main result of this section reads as follows.

Theorem 7.1 (Energetic solvability of the quasistatic system). *Assume polyconvexity of W_e and coercivity of W_e and \widehat{W}_p , namely*

$$\begin{aligned} W_e(\mathbf{F}_e) &= \mathbb{W}(\mathbf{F}_e, \text{cof } \mathbf{F}_e, \det \mathbf{F}_e) \quad \forall \mathbf{F}_e \in \text{GL}^+ \\ \text{for some } \mathbb{W} : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times \mathbb{R}_+ &\rightarrow \mathbb{R} \quad \text{convex,} \end{aligned} \quad (7.1)$$

$$W_e(\mathbf{F}_e) \geq c_4 |\mathbf{F}_e|^{q_e} - \frac{1}{c_4}, \quad \widehat{W}_p(\mathbf{C}_p) \geq c_4 |\mathbf{C}_p|^{q_p} - \frac{1}{c_4} \quad (7.2)$$

for some positive constant c_4 , and $q_e, q_p > 1$. Moreover, assume that

$$\frac{1}{q_y} = \frac{1}{q_e} + \frac{1}{2q_p}, \quad q_y > 3, \quad r > 1. \quad (7.3)$$

Eventually, let $\ell \in C^1([0, T]; (W_{\Gamma_D}^{1,q_y}(\Omega; \mathbb{R}^3))^*)$ and $(y_0, \mathbf{C}_{p,0}) \in \mathcal{S}(0)$ where $\mathcal{S}(t)$ denotes the set of stable states at time $t \in [0, T]$ with respect to $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ at time t . Then, there exists an energetic solution corresponding to $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ starting from $(y_0, \mathbf{C}_{p,0})$. More precisely, for all partitions $\{0 = t_0^k < t_1^k < \dots < t_{N^k}^k = T\}$ with time step $\tau^k = \max(t_i^k - t_{i-1}^k)$ the incremental minimization problems

$$\begin{aligned} (y_i, \mathbf{C}_{p,i}) &= \text{Argmin} \{ \mathcal{E}(y, \mathbf{C}_p, t_i^k) + \mathcal{D}(\mathbf{C}_{p,i-1}, \mathbf{C}_{p,i}) \mid (y, \mathbf{C}_p) \in \mathcal{Q} \} \\ &\text{for } i = 1, \dots, N^k \end{aligned}$$

admit a solution $\{(y_0, \mathbf{C}_{p,0}), (y_1^k, \mathbf{C}_{p,1}^k), \dots, (y_{N^k}^k, \mathbf{C}_{p,N^k}^k)\}$ and, as $\tau^k \rightarrow 0$, the corresponding piecewise backward-constant interpolants $t \mapsto (\bar{y}^k(t), \bar{\mathbf{C}}_p^k(t))$ on the partition admit a not relabeled subsequence such that, for all $t \in [0, T]$,

$$\begin{aligned} (\bar{y}^k(t), \bar{\mathbf{C}}_p^k(t)) &\rightarrow (y(t), \mathbf{C}_p(t)), \quad \text{Diss}_{\mathcal{D},[0,t]}(\bar{\mathbf{C}}_p^k) \rightarrow \text{Diss}_{\mathcal{D},[0,t]}(\mathbf{C}_p), \\ \mathcal{E}(\bar{y}^k(t), \bar{\mathbf{C}}_p^k(t), t) &\rightarrow \mathcal{E}(y(t), \mathbf{C}_p(t), t), \end{aligned}$$

and $\partial_t \mathcal{E}(\bar{y}^k(\cdot), \bar{\mathbf{C}}_p^k(\cdot), \cdot) \rightarrow \partial_t \mathcal{E}(y(t), \mathbf{C}_p(\cdot), \cdot)$ in $L^1(0, T)$ where (y, \mathbf{C}_p) is an energetic solution corresponding to $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$.

Before moving on let us comment that an elastic energy density W_e satisfying (7.1)-(7.2) can be found, for instance, within the class of *Ogden materials* [9, Sec. 4.9] corresponding indeed to the choice

$$\begin{aligned} W_e(\mathbf{F}_e) &:= \widehat{W}_e(\mathbf{C}_e) := \sum_{i=1}^n a_i \text{tr} \mathbf{C}_e^{\gamma_i/2} + \sum_{j=1}^m b_j \text{tr} (\text{cof} \mathbf{C}_e)^{\delta_j/2} + \Gamma(\det \mathbf{C}_e^{1/2}), \\ n, m &\geq 1, \quad a_i, b_j > 0, \quad \gamma_i, \delta_j \geq 1, \quad s \mapsto \Gamma(s) \text{ convex on } (0, \infty), \quad \lim_{s \rightarrow 0^+} \Gamma(s) = \infty. \end{aligned}$$

Clearly, frame indifference and isotropy are fulfilled. Moreover the function $W_e(\mathbf{F}_e)$ is polyconvex as $\gamma_i, \delta_j \geq 1$. Note that $\gamma_i \geq 1$ for some i implies $q_e \geq 2$ in condition (7.2).

We shall now prepare some lemmas to be used in the proof of Theorem 7.1.

Lemma 7.2 (Coercivity of the energy). *Under the assumptions of Theorem 7.1, the energy \mathcal{E} is coercive in the following sense*

$$\mathcal{E}(y, \mathbf{C}_p, t) \geq c_5 \|y\|_{W^{1,q_y}}^{q_y} + c_5 \|\mathbf{C}_p\|_{L^{q_p}}^{q_p} - \frac{1}{c_5} \quad (7.4)$$

where c_5 is a positive constant.

Proof. From the coercivity assumption (7.2), an application of Young's inequality gives

$$\begin{aligned} \frac{1}{c_4} W_e(\mathbf{F}_e) + \frac{1}{c_4^2} &\geq |\nabla y \mathbf{C}_p^{-1/2}|^{q_e} \geq (|\nabla y|/|\mathbf{C}_p^{1/2}|)^{q_e} \geq 3^{-1/2} (|\nabla y|^{q_e}/|\mathbf{C}_p|^{q_e/2}) \geq \\ &3^{-1/2} ((1+t)\delta^{t/(t+1)} |\nabla y|^{q_e/(t+1)} - t\delta |\mathbf{C}_p|^{q_e/2t}) \end{aligned}$$

for any δ and t positive. Moreover, $\widehat{W}_p(\mathbf{C}_p) \geq c_4 |\mathbf{C}_p|^{q_p} - 1/c_4$. By taking δ sufficiently small and t such that $q_e/2t = q_p$, we obtain

$$W_e(\mathbf{F}_e) + \widehat{W}_p(\mathbf{C}_p) \geq c |\nabla y|^{2q_e q_p/(2q_p+q_e)} + c |\mathbf{C}_p|^{q_p} - c,$$

therefore

$$\mathcal{W}(y, \mathbf{C}_p) \geq c \|\nabla y\|_{L^{q_y}}^{q_y} + \|\mathbf{C}_p\|_{L^{q_p}}^{q_p} - c.$$

As $|\langle \ell, y \rangle| \leq \|\ell\| \|y\|_{W^{1,q_y}}$, by virtue of Korn's inequality the statement follows. \square

Note that indeed the coercivity lower bound (7.4) holds under the weaker condition $1/q_y = 1/q_e + 1/(2q_p) < 1$ as well.

We shall now prove the weak semicontinuity of the energy. To establish it, the following Lemma 7.3 on the convergence of minors is needed. We recall that for all $\mathbf{A} \in \mathbb{R}^{3 \times 3}$, we have a 3×3 matrix of order-one minors: $\mathbb{M}_1(\mathbf{A}) = \mathbf{A}$, a 3×3 matrix of order-two minors: $\mathbb{M}_2(\mathbf{A}) = \text{cof} \mathbf{A}$, and a scalar minor of order three $\mathbb{M}_3(\mathbf{A}) = \det \mathbf{A}$. By introducing the shorthand notation $\mathbb{M}(\mathbf{A}) = (\mathbb{M}_1(\mathbf{A}), \mathbb{M}_2(\mathbf{A}), \mathbb{M}_3(\mathbf{A}))$, according to assumption (7.1) we can write $W_e(\nabla y \mathbf{C}_p^{-1/2}) = \mathbb{W}(\mathbb{M}(\nabla y \mathbf{C}_p^{-1/2}))$. The next Lemma is a particular case of [43, Prop. 5.1].

Lemma 7.3 (Convergence of minors). *Let $y_k \rightharpoonup y$ in $W^{1,q_y}(\Omega; \mathbb{R}^3)$ and $\mathbf{P}_k \rightarrow \mathbf{P}$ in $L^p(\Omega; \text{SL})$ and*

$$q_y > 3, \quad \frac{1}{q_y} + \frac{2}{p} \leq 1. \quad (7.5)$$

Then,

$$\mathbb{M}(\nabla y_k \mathbf{P}_k^{-1}) \rightharpoonup \mathbb{M}(\nabla y \mathbf{P}^{-1}) \text{ in } L^1(\Omega; \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times \mathbb{R}).$$

Proof. Since $\det \mathbf{P}_k = 1$, we have

$$\mathbb{M}(\nabla y_k \mathbf{P}_k^{-1}) = \left(\nabla y_k (\text{cof} \mathbf{P}_k)^\top, \text{cof}(\nabla y_k) \mathbf{P}_k^\top, \det(\nabla y_k) \right). \quad (7.6)$$

The desired convergence is obtained from the fact that

$$\frac{1}{\rho} + \frac{1}{\sigma} \leq 1, \quad \mathbf{A}_k \xrightarrow{L^\rho} \mathbf{A}, \quad \mathbf{B}_k \xrightarrow{L^\sigma} \mathbf{B} \Rightarrow \mathbf{A}_k \mathbf{B}_k \xrightarrow{L^1} \mathbf{A} \mathbf{B},$$

applied to the three minors in (7.6). The classical weak continuity of gradient minors [4] for $q_y > 3$ and $y_k \rightharpoonup y$ in W^{1,q_y} yield

$$\nabla y_k \xrightarrow{L^{q_y}} \nabla y, \quad \text{cof}(\nabla y_k) \xrightarrow{L^{q_y/2}} \text{cof}(\nabla y), \quad \det(\nabla y_k) \xrightarrow{L^{q_y/3}} \det(\nabla y).$$

Moreover, we clearly have that

$$\mathbf{P}_k^\top \xrightarrow{L^p} \mathbf{P}^\top, \quad \text{cof} \mathbf{P}_k \xrightarrow{L^{p/2}} \text{cof} \mathbf{P}.$$

In order to conclude, the following conditions on indexes need to be checked

$$q_y > 3, \quad \frac{1}{q_y} + \frac{2}{p} \leq 1, \quad \frac{2}{q_y} + \frac{1}{p} \leq 1.$$

The first two are (7.5) and the last one is a direct consequence of these. \square

We can now establish the weak lower semicontinuity of the energy functional.

Lemma 7.4 (Lower semicontinuity of the energy). *Under the assumptions of Theorem 7.1 the energy \mathcal{E} is weakly lower semicontinuous.*

Proof. Let $(y_k, \mathbf{C}_{pk}) \rightharpoonup (y, \mathbf{C}_p)$ in \mathcal{Q} . The compact embedding of $W^{1,r} \subset\subset L^r$ and the weak convergence of $\mathbf{C}_{pk} \rightharpoonup \mathbf{C}_p$ in $L^{q_p} \cap W^{1,r}$ entail strong convergence in L^s , for all $s \in [1, q_p)$ if $r < q_p$ and all $s \in [1, r]$ if $r \geq q_p$. The term

$$\mathbf{C}_p \mapsto \int_{\Omega} \left(\widehat{W}_p(\mathbf{C}_p) + \frac{\mu}{r} |\nabla \mathbf{C}_p|^r \right) dx$$

is hence lower semicontinuous (see for example [68, Thm. 1.6, p. 9]).

The convergence in measure of \mathbf{C}_{pk} implies the convergence in measure of $\mathbf{C}_{pk}^{1/2}$. This follows from the local Lipschitz continuity result in Lemma A.1. Convergence in measure and the L^{2q_p} boundedness of $\mathbf{C}_{pk}^{1/2}$, then yield the strong convergence $\mathbf{C}_{pk}^{1/2} \rightarrow \mathbf{C}_p^{1/2}$ in L^p for $p \in [1, 2q_p)$. Thanks to Lemma 7.3, we have

$$\mathbb{M}(\nabla y_k \mathbf{C}_{pk}^{-1/2}) \rightharpoonup \mathbb{M}(\nabla y \mathbf{C}_p^{-1/2}) \text{ in } L^1.$$

In fact, condition (7.5) is easily verified by checking that

$$\frac{1}{q_y} + \frac{2}{2q_p} \stackrel{(7.3)}{=} \frac{1}{q_e} + \frac{3}{2q_p} \stackrel{(7.3)}{<} 1 - \frac{2}{q_e},$$

which implies $1/q_y + 2/p \leq 1$ for $p = 2q_p - \varepsilon \in [1, 2q_p)$ and $\varepsilon > 0$ sufficiently small. As $W_e(\nabla y \mathbf{C}_p^{-1/2}) = \mathbb{W}(\mathbb{M}(\nabla y \mathbf{C}_p^{-1/2}))$, with \mathbb{W} convex, the lower semicontinuity of the elastic energy term follows. Eventually, the time-dependent linear term is weakly continuous. \square

We are now ready to prove the existence of energetic solution for the quasistatic evolution problem.

Proof of Theorem 7.1. We aim at checking the assumptions of the abstract existence Lemma 3.1. The structural conditions (3.4) follow directly from the properties of the density D . Thanks to Lemmas 7.2 and 7.4, the sublevels of energy are bounded and weakly closed in \mathcal{Q} , whence the compactness follows. The power reads

$$\partial_t \mathcal{E}(q, t) = -\langle \dot{\ell}, y \rangle, \quad \dot{\ell} \in C^0([0, T]; (W_{\Gamma_D}^{1,q_y}(\Omega, \mathbb{R}^3))^*).$$

In particular, assumption (3.5) follows. Eventually, the closure of stable states (3.6) is a consequence of the lower semicontinuity of the energy \mathcal{E} and the continuity of the dissipation \mathcal{D} . \square

Before moving on, let us remark that the weaker assumption

$$\ell \in W^{1,1}([0, T]; (W_{\Gamma_D}^{1,q_y}(\Omega, \mathbb{R}^3))^*)$$

on external actions would also suffice to prove an existence result. In this case, the abstract result of Lemma 3.1 has to be adapted and indeed simplified to the case of linear applied loads, see for instance [2].

8. SMALL-DEFORMATION LIMIT FOR QUASISTATIC EVOLUTION

Let us now turn to the proof of a linearization result for quasistatic evolutions. The aim here is to present the three-dimensional version of the main result of Section 5, namely Theorem 5.1. Again, this consists in a variational convergence argument.

In the stationary, multidimensional framework, the seminal contribution in this respect is [13] where a variational justification of linearization in elasticity is provided. Successive refinements [1] and extensions [57, 58, 63] of the argument have been presented. In the evolutive case, the first result in plasticity with hardening is in [50]. Linearized plate models have been derived from finite plasticity in [14, 15] and perfect plasticity is considered in [27]. We follow here the general strategy of [50], by adapting indeed many technical points to the present symmetric situation. In particular, the choice of rescaled variables and functionals is here different from [50] and especially tailored to cope with the nonlinear structure of SL_{sym}^+ . Note additionally that [50] proves the convergence of finite-plasticity trajectories, whose existence under the assumptions of that paper is not known. The situation is here different as energetic solutions do exist under the assumptions of Theorem 7.1. The price to pay here is that we have to discuss the limiting behavior of the gradient term in $\nabla \mathbf{C}_p$. In particular, our small-deformation analysis is restricted to the case $r = 2$ in (6.5) for the quadratic character of the gradient energy term plays a crucial role.

Letting $\varepsilon > 0$ and $(y, \mathbf{C}_p) \in \mathcal{Q}$ be given we introduce the equivalent variables

$$u = \frac{1}{\varepsilon}(y - \text{id}), \quad \mathbf{z} = \frac{1}{2\varepsilon} \log \mathbf{C}_p \quad (8.1)$$

as well as the rescaled functionals for $r = 2$

$$\begin{aligned} \mathcal{E}_\varepsilon(u, \mathbf{z}, t) &= \int_{\Omega} W_\varepsilon(\mathbf{e}_\varepsilon, \mathbf{z}) \, dx + \frac{\mu}{2\varepsilon^2} \int_{\Omega} |\nabla \exp(2\varepsilon \mathbf{z})|^2 - \langle \ell(t), u \rangle, \\ \mathcal{D}_\varepsilon(\mathbf{z}_1, \mathbf{z}_2) &= \int_{\Omega} D_\varepsilon(\mathbf{z}_1, \mathbf{z}_2) \, dx \end{aligned}$$

where the rescaled Green-Saint Venant strain \mathbf{e}_ε is given by

$$\mathbf{e}_\varepsilon = \frac{1}{2\varepsilon} ((\mathbf{I} + \varepsilon \nabla u)^\top (\mathbf{I} + \varepsilon \nabla u) - \mathbf{I}) = \nabla u^{\text{sym}} + \frac{\varepsilon}{2} \nabla u^\top \nabla u.$$

Assume now to be given $\ell \in C^1([0, T]; (W_{\Gamma_D}^{1, q_y}(\Omega; \mathbb{R}^3)))^*$ and initial values $\mathbf{z}_{0\varepsilon}$ such that $\exp(2\varepsilon \mathbf{z}_{0\varepsilon}) \in \mathcal{S}(0)$ where $\mathcal{S}(t)$ denotes stable states at time $t \in [0, T]$ corresponding to $(\mathcal{Q}, \mathcal{E}/\varepsilon^2, \mathcal{D}/(2\varepsilon))$. Owing to Theorem 7.1 there exists an energetic solution $(y_\varepsilon, \mathbf{C}_{p\varepsilon})$ corresponding to $(\mathcal{Q}, \mathcal{E}/\varepsilon^2, \mathcal{D}/(2\varepsilon))$. Correspondingly, by defining $(u_\varepsilon, \mathbf{z}_\varepsilon)$ from $(y_\varepsilon, \mathbf{C}_{p\varepsilon})$ via (8.1) we readily find that $(u_\varepsilon, \mathbf{z}_\varepsilon)$ is an energetic solution corresponding to $(\mathcal{Q}_0, \mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)$ where the space \mathcal{Q}_0 can be chosen as

$$\mathcal{Q}_0 = H_{\Gamma_D}^1(\Omega; \mathbb{R}^3) \times H^1(\Omega; \mathbb{R}^{3 \times 3}_{\text{dev}})$$

by simply extending trivially the functionals. We shall refer to $(u_\varepsilon, \mathbf{z}_\varepsilon)$ as *finite-plasticity quasistatic evolutions* and denote the corresponding set of stable states at time $t \in [0, T]$ by $\mathcal{S}_\varepsilon(t)$.

We are here concerned with the convergence of finite-plasticity quasistatic evolutions $(u_\varepsilon, \mathbf{z}_\varepsilon)$ to the unique solution (u, \mathbf{z}) of the linearized elastoplasticity system corresponding to $(\mathcal{Q}_0, \mathcal{E}_0, \mathcal{D}_0)$ where

$$\begin{aligned}\mathcal{E}_0(u, \mathbf{z}, t) &= \int_{\Omega} W_0(\nabla u^{\text{sym}}, \mathbf{z}) \, dx + 2\mu \int_{\Omega} |\nabla \mathbf{z}|^2 - \langle \ell(t), u \rangle, \\ \mathcal{D}_0(\mathbf{z}, \widehat{\mathbf{z}}) &= \int_{\Omega} D_0(\mathbf{z}, \widehat{\mathbf{z}}) \, dx = \int_{\Omega} r |\mathbf{z}_1 - \mathbf{z}_2| \, dx.\end{aligned}$$

In particular, (u, \mathbf{z}) is the unique strong solution of the equilibrium system (6.1) with $\boldsymbol{\sigma} = \mathbb{C}(\nabla u^{\text{sym}} - \mathbf{z})$ and boundary conditions (6.2)-(6.3), and of the constitutive relation

$$R\partial|\dot{\mathbf{z}}| + (\mathbb{H} + \mathbb{C})\mathbf{z} - 4\mu\Delta\mathbf{z} \ni \mathbb{C}\nabla u^{\text{sym}}$$

along with the homogeneous Neumann condition $\mu\nabla\mathbf{z}\nu = \mathbf{0}$ and an initial condition for \mathbf{z} [31]. We refer to (u, \mathbf{z}) as *linearized-plasticity quasistatic evolution*.

We now state our main convergence result.

Theorem 8.1 (Small-deformation limit of the quasistatic evolution). *Let the compact set*

$$K := \{\mathbf{C} \in \text{SL}_{\text{sym}}^+ : |\mathbf{C}| \leq r\}, \quad r^2 > 3 \quad (8.2)$$

be given and assume \widehat{W}_p to be coercive in the following sense

$$\widehat{W}_p(\mathbf{C}_p) < \infty \Leftrightarrow \mathbf{C}_p, \mathbf{C}_p^{-1} \in K. \quad (8.3)$$

Moreover, assume the control (4.5), the quadratic behavior at identity (5.4), and let

$$W_e(\mathbf{F}) \geq c_6 \text{dist}^2(\mathbf{F}, \text{SO}) \quad \forall \mathbf{F} \in \text{GL}^+ \quad (8.4)$$

for some positive constant c_6 . Let $(u_\varepsilon, \mathbf{z}_\varepsilon)$ be finite-plasticity quasistatic evolutions starting from well-prepared initial data $(u_{0\varepsilon}, \mathbf{z}_{0\varepsilon}) \in \mathcal{S}_\varepsilon(0)$, namely,

$$(u_{0\varepsilon}, \mathbf{z}_{0\varepsilon}) \rightarrow (u_0, \mathbf{z}_0) \quad \text{and} \quad \mathcal{E}_\varepsilon(u_{0\varepsilon}, \mathbf{z}_{0\varepsilon}, 0) \rightarrow \mathcal{E}_0(u_0, \mathbf{z}_0, 0).$$

Then, for all $t \in [0, T]$,

$$\begin{aligned}(u_\varepsilon(t), \mathbf{z}_\varepsilon(t)) &\rightarrow (u(t), \mathbf{z}(t)), \\ \text{Diss}_{\mathcal{D}_\varepsilon, [0, t]}(\mathbf{z}_\varepsilon) &\rightarrow \text{Diss}_{\mathcal{D}_0, [0, t]}(\mathbf{z}), \\ \mathcal{E}_\varepsilon(u_\varepsilon(t), \mathbf{z}_\varepsilon(t), t) &\rightarrow \mathcal{E}_0(u(t), \mathbf{z}(t), t)\end{aligned}$$

where (u, \mathbf{z}) is the unique linearized-plasticity quasistatic evolution starting from (u_0, \mathbf{z}_0) .

The coercivity assumption on \widehat{W}_p is stronger than the former (4.7), (5.3), and (7.2). The assumption on the shape of K from (8.2) is of technical nature and could probably be relaxed. Let us however stress that it arises quite naturally in modeling pseudoplastic

processes in shape memory alloys [20, 21, 59]. In addition, note that the choice for the shape of K is immaterial with respect to the linearization limit $\varepsilon \rightarrow 0$.

Let us start by presenting a coercivity result.

Lemma 8.2 (Coercivity of the energy). *Under the assumptions of Theorem 8.1 we have that*

$$\|\nabla u\|_{L^2}^2 + \|\mathbf{z}\|_{L^2}^2 + \|\nabla \exp(2\varepsilon \mathbf{z})/\varepsilon\|_{L^2}^2 + \varepsilon \|\mathbf{z}\|_{L^\infty} \leq c_7(1 + \mathcal{W}_\varepsilon(u, \mathbf{z})) \quad (8.5)$$

for all $(u, \mathbf{z}) \in \mathcal{Q}$ where c_7 is a positive constant.

Proof. Assume $\mathcal{W}_\varepsilon(u, \mathbf{z}) < \infty$. Then, $\exp(2\varepsilon \mathbf{z}) \in K$ so that $\varepsilon \|\mathbf{z}\|_{L^\infty} \leq c$. In fact, by denoting by λ_i the eigenvalues of \mathbf{z} , condition $\text{tr } \mathbf{z} = 0$ entails $\lambda_{\max} \geq \|\mathbf{z}\|_\infty/6$, hence $\|\exp(2\varepsilon \mathbf{z})\|_\infty \geq e^{2\varepsilon \lambda_{\max}} \geq e^{\varepsilon \|\mathbf{z}\|_{L^\infty}/3}$. From the coercivity of \widehat{W}_p we have that $\|\mathbf{z}\|_{L^2} + \|\nabla \exp(2\varepsilon \mathbf{z})/\varepsilon\|_{L^2} \leq c\mathcal{W}_\varepsilon(u, \mathbf{z})$. Note that, for all $\alpha \in \mathbb{R}$

$$\exp(\alpha \varepsilon \mathbf{z}) = \mathbf{I} + \alpha \varepsilon \mathbf{z} + \varepsilon^2 \mathbf{z}^2 \mathbf{B}_\varepsilon, \quad \mathbf{B}_\varepsilon := \left(\sum_{k=2}^{\infty} \frac{\alpha^k (\varepsilon \mathbf{z})^{k-2}}{k!} \right). \quad (8.6)$$

As $\|\varepsilon \mathbf{z}\|_{L^\infty} \leq c$, we have that $\|\mathbf{B}_\varepsilon\|_{L^\infty} \leq c$.

The coercivity estimate on ∇u is based on a geometric rigidity argument [25] as in [13]. The first step is to obtain an estimate of the distance of ∇y from SO. By using $\mathbf{F}_e := \nabla y \mathbf{C}_p^{-1/2}$ and, recalling that $\mathbf{C}_p \in K$ is bounded, one obtains

$$|\nabla y - \mathbf{Q}|^2 = |(\mathbf{F}_e - \mathbf{Q})\mathbf{C}_p^{1/2} + \mathbf{Q}(\mathbf{C}_p^{1/2} - \mathbf{I})|^2 \leq c|\mathbf{F}_e - \mathbf{Q}|^2 + |\mathbf{C}_p^{1/2} - \mathbf{I}|^2$$

almost everywhere. We now use (8.6) with $\alpha = 1$ on order to get that $\mathbf{C}_p^{1/2} = \mathbf{I} + \varepsilon \mathbf{z} + \varepsilon^2 \mathbf{z}^2 \mathbf{B}_\varepsilon = \mathbf{I} + \varepsilon \mathbf{z} \mathbf{B}'_\varepsilon$, where $\mathbf{B}'_\varepsilon = \mathbf{I} + (\varepsilon \mathbf{z}) \mathbf{B}_\varepsilon$. Therefore

$$|\mathbf{C}_p^{1/2} - \mathbf{I}|^2 \leq \varepsilon^2 |\mathbf{z}|^2 |\mathbf{B}'_\varepsilon|^2$$

and we conclude for

$$|\nabla y - \mathbf{Q}|^2 = |(\mathbf{F}_e - \mathbf{Q})\mathbf{C}_p^{1/2} + \mathbf{Q}(\mathbf{C}_p^{1/2} - \mathbf{I})|^2 \leq c(|\mathbf{F}_e - \mathbf{Q}|^2 + \varepsilon^2 |\mathbf{z}|^2 |\mathbf{B}'_\varepsilon|^2)$$

where $\|\mathbf{B}'_\varepsilon\|_{L^\infty} \leq c$. We now proceed as in [50, Lemma 3.1]. The last inequality combined with the nondegeneracy condition (8.4) yields

$$\int_{\Omega} \text{dist}^2(\nabla y, \text{SO}) \, dx \leq c\varepsilon^2(1 + \mathcal{W}_\varepsilon(u, \mathbf{z})).$$

Then, the Rigidity Lemma [25, Thm. 3.1] entails

$$\exists \widehat{\mathbf{Q}} \in \text{SO}(3) : \quad \|\nabla y - \widehat{\mathbf{Q}}\|_{L^2}^2 \leq c\varepsilon^2(1 + \mathcal{W}_\varepsilon(u, \mathbf{z})).$$

As the rotation $\widehat{\mathbf{Q}}$ satisfies the estimate $|\widehat{\mathbf{Q}} - \mathbf{I}|^2 \leq c\varepsilon^2(1 + \mathcal{W}_\varepsilon(u, \mathbf{z}))$ as a result of the boundary conditions, see [13, Prop. 3.4], we have

$$\varepsilon^2 \|\nabla u\|_{L^2}^2 = \|\nabla y - \mathbf{I}\|_{L^2}^2 \leq 2\|\nabla y - \widehat{\mathbf{Q}}\|_{L^2}^2 + 2\|\widehat{\mathbf{Q}} - \mathbf{I}\|_{L^2}^2 \leq c\varepsilon^2(1 + \mathcal{W}_\varepsilon(u, \mathbf{z}))$$

and the assertion follows. \square

The next step toward to application of the evolutive Γ -convergence Lemma 3.2 is the proof of the Γ -lim inf inequalities (3.7)-(3.8). We do this in the next two lemmas.

Lemma 8.3 (Γ -lim inf inequality for \mathcal{E}_ε). *Under the assumptions of Theorem 8.1 we have that*

$$\mathcal{E}_0(u, \mathbf{z}, t) \leq \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon, \mathbf{z}_\varepsilon, t_\varepsilon) \mid (u_\varepsilon, \mathbf{z}_\varepsilon, t_\varepsilon) \xrightarrow{\mathcal{Q} \times [0, T]} (u, \mathbf{z}, t) \right\}.$$

Proof. As convergence for the linear external energy $\langle \ell(t_\varepsilon), u_\varepsilon \rangle$ is trivial, we concentrate on the terms

$$\begin{aligned} I_\varepsilon^1 &:= \frac{1}{\varepsilon^2} \int_{\Omega} \widehat{W}_e(\exp(-\varepsilon \mathbf{z}_\varepsilon)(\mathbf{I} + 2\varepsilon \mathbf{e}_\varepsilon) \exp(-\varepsilon \mathbf{z}_\varepsilon)) \, dx, \\ &= \frac{1}{\varepsilon^2} \int_{\Omega} W_e((\mathbf{I} + \varepsilon \nabla u_\varepsilon) \exp(-\varepsilon \mathbf{z}_\varepsilon)) \, dx \\ I_\varepsilon^2 &:= \frac{1}{\varepsilon^2} \int_{\Omega} \widehat{W}_p(\exp(2\varepsilon \mathbf{z}_\varepsilon)) \, dx, \\ I_\varepsilon^3 &:= \frac{\mu}{2\varepsilon^2} \int_{\Omega} |\nabla \exp(2\varepsilon \mathbf{z}_\varepsilon)|^2 \, dx = 2\mu \int_{\Omega} |\nabla \exp(2\varepsilon \mathbf{z}_\varepsilon)/(2\varepsilon)|^2 \, dx. \end{aligned}$$

Let $(u_\varepsilon, \mathbf{z}_\varepsilon) \xrightarrow{\mathcal{Q}} (u, \mathbf{z})$. We can assume with no loss of generality that $\sup \mathcal{E}_\varepsilon(u_\varepsilon, \mathbf{z}_\varepsilon, t_\varepsilon) < \infty$, so that the bound (8.5) holds and $\exp(2\varepsilon \mathbf{z}_\varepsilon) \in K$. In particular, $\|\varepsilon \mathbf{z}_\varepsilon\|_{L^\infty} \leq c$.

Relation (8.6) for $\alpha = 2$ entails that $\exp(2\varepsilon \mathbf{z}_\varepsilon) = \mathbf{I} + 2\varepsilon \mathbf{z}_\varepsilon + \varepsilon^2 \mathbf{z}_\varepsilon^2 \mathbf{B}_\varepsilon$ with $\|\mathbf{B}_\varepsilon\|_{L^\infty} \leq c$. As $\|\mathbf{z}_\varepsilon\|_{L^2} \leq c$, we compute

$$\frac{1}{2\varepsilon} \left(\exp(2\varepsilon \mathbf{z}_\varepsilon) - \mathbf{I} \right) - \mathbf{z}_\varepsilon = \frac{1}{2} \varepsilon \mathbf{z}_\varepsilon^2 \mathbf{B}_\varepsilon \xrightarrow{L^1} 0$$

and, taking into account the L^2 -boundedness of the same sequence, we have checked that $(\exp(2\varepsilon \mathbf{z}_\varepsilon) - \mathbf{I})/(2\varepsilon) \rightharpoonup \mathbf{z}$ in L^2 . On the other hand, by (8.5) we also have the gradient bound $\|\nabla(\exp(2\varepsilon \mathbf{z}_\varepsilon) - \mathbf{I})/(2\varepsilon)\|_{L^2} \leq c$. Hence, convergence

$$\frac{1}{2\varepsilon} \left(\exp(2\varepsilon \mathbf{z}_\varepsilon) - \mathbf{I} \right) \xrightarrow{H^1} \mathbf{z} \tag{8.7}$$

follows. In particular,

$$2\mu \int_{\Omega} |\nabla \mathbf{z}|^2 \, dx \leq \liminf_{\varepsilon \rightarrow 0} 2\mu \int_{\Omega} |\nabla \exp(2\varepsilon \mathbf{z}_\varepsilon)/(2\varepsilon)|^2 \, dx = \liminf_{\varepsilon \rightarrow 0} I_\varepsilon^3.$$

Convergence (5.9) and Lemma C.1 directly entail that

$$\frac{1}{2} \int_{\Omega} |\mathbf{z}|_{\mathbb{H}}^2 \, dx \leq \liminf_{\varepsilon \rightarrow 0} I_\varepsilon^2.$$

Let us now turn to integral I_ε^1 . We introduce, also for future reference, the shorthand notation

$$\mathbf{A}_\varepsilon = \frac{1}{\varepsilon} \left((\mathbf{I} + \varepsilon \nabla u_\varepsilon) \exp(-\varepsilon \mathbf{z}_\varepsilon) - \mathbf{I} \right), \tag{8.8}$$

which allows to rewrite I_ε^1 as

$$I_\varepsilon^1 = \frac{1}{\varepsilon^2} \int_{\Omega} W_e(1 + \varepsilon \mathbf{A}_\varepsilon) dx.$$

In view of convergence (5.8) and Lemma C.1, it is sufficient to prove

$$\mathbf{A}_\varepsilon \xrightarrow{L^2} \nabla u - \mathbf{z}. \quad (8.9)$$

By expanding $\exp(-\varepsilon \mathbf{z}_\varepsilon) = \mathbf{I} - \varepsilon \mathbf{z}_\varepsilon + \varepsilon^2 \mathbf{z}_\varepsilon^2 \mathbf{B}_\varepsilon$ according to (8.6) with $\alpha = -1$ we compute

$$\mathbf{A}_\varepsilon = (\nabla u_\varepsilon - \mathbf{z}) + (\mathbf{z} - \mathbf{z}_\varepsilon) + \varepsilon \mathbf{z}_\varepsilon^2 \mathbf{B}_\varepsilon + \varepsilon \nabla u_\varepsilon \mathbf{z}_\varepsilon (\varepsilon \mathbf{z}_\varepsilon \mathbf{B}_\varepsilon - \mathbf{I}).$$

As $\nabla u_\varepsilon \rightharpoonup \nabla u$ in L^2 , $\mathbf{z}_\varepsilon \rightharpoonup \mathbf{z}$ in L^2 , and $\|\mathbf{B}_\varepsilon\|_{L^\infty} + \|\varepsilon \mathbf{z}_\varepsilon\|_{L^\infty} \leq c$, the convergence (8.9) follows. \square

Lemma 8.4 (Γ -lim inf inequality for \mathcal{D}_ε). *Under the assumptions of Theorem 8.1 we have that*

$$\mathcal{D}_0(\mathbf{z}, \widehat{\mathbf{z}}) \leq \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \mathcal{D}_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \mid (\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \rightharpoonup (\mathbf{z}, \widehat{\mathbf{z}}) \text{ in } L^2(\Omega; (\mathbb{R}_{\text{dev}}^{3 \times 3})^2) \right\}.$$

Proof. The assertion follows by Lemma 5.3 by applying the lower semicontinuity tool of Lemma C.1. \square

Having established the above Γ -lim inf inequalities, the next ingredient of the evolutive- Γ -convergence argument is the specification of a *mutual recovery sequence*. This is done within the following lemma.

Lemma 8.5 (Mutual recovery sequence). *Under the assumptions of Theorem 8.1 let $(u_\varepsilon, \mathbf{z}_\varepsilon) \xrightarrow{\mathcal{Q}} (u_0, \mathbf{z}_0)$ be given with $\sup_\varepsilon \mathcal{E}_\varepsilon(t, u_\varepsilon, \mathbf{z}_\varepsilon) < \infty$. Moreover, let*

$$(\widehat{u}_0, \widehat{\mathbf{b}}_{\mathbf{z}_0}) = (u_0, \mathbf{z}_0) + (\widetilde{u}, \widetilde{\mathbf{z}})$$

where $(\widetilde{u}, \widetilde{\mathbf{z}}) \in C_c^\infty(\Omega; \mathbb{R}^3) \times C_c^\infty(\Omega; \mathbb{R}_{\text{dev}}^{3 \times 3})$. Then, there exists $(\widehat{u}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \in \mathcal{Q}$ such that $(\widehat{u}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \xrightarrow{\mathcal{Q}} (\widehat{u}_0, \widehat{\mathbf{z}}_0)$ and

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{D}_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \leq \mathcal{D}_0(\mathbf{z}_0, \widehat{\mathbf{z}}_0) = R(\widetilde{\mathbf{z}}), \quad (8.10)$$

$$\limsup_{\varepsilon \rightarrow 0} \left(\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon, t) - \mathcal{E}_\varepsilon(u_\varepsilon, \mathbf{z}_\varepsilon, t) \right) \leq \mathcal{E}_0(\widehat{u}_0, \widehat{\mathbf{z}}_0, t) - \mathcal{E}_0(u_0, \mathbf{z}_0, t). \quad (8.11)$$

Proof. We divide the proof into several steps.

Step 1: Definition of the recovery sequence. We define

$$\widehat{u}_\varepsilon := u_\varepsilon + \widetilde{u} \circ (\text{id} + \varepsilon u_\varepsilon) \quad (8.12)$$

$$\widehat{\mathbf{z}}_\varepsilon := \frac{1}{2\varepsilon} \log \left(\mathbf{\Pi} \left(\exp(2\varepsilon(\mathbf{z}_\varepsilon + \widetilde{\mathbf{z}})) \right) \right) \quad (8.13)$$

where $\mathbf{\Pi} : \text{SL}_{\text{sym}}^+ \rightarrow K$ is a contraction mapping onto the compact set K defined by (8.2). More precisely, we ask $\mathbf{\Pi}$ to have the following properties of $\mathbf{\Pi}$:

$$\mathbf{\Pi}|_K = \text{id}_K \quad \text{and} \quad |\mathbf{\Pi}(\mathbf{C}_{p1}) - \mathbf{\Pi}(\mathbf{C}_{p2})| \leq |\mathbf{C}_{p1} - \mathbf{C}_{p2}| \quad \forall \mathbf{C}_{p1}, \mathbf{C}_{p2} \in \text{SL}_{\text{sym}}^+. \quad (8.14)$$

An explicit construction of a map $\mathbf{\Pi}$ fulfilling these properties is provided in Appendix B.

The definition of \widehat{u}_ε can be rewritten in terms of $\widehat{y}_\varepsilon = \text{id} + \varepsilon \widehat{u}_\varepsilon$ as

$$\widehat{y}_\varepsilon = \text{id} + \varepsilon u_\varepsilon + \varepsilon \widetilde{u} \circ (\text{id} + \varepsilon u_\varepsilon) = \widetilde{y} \circ y_\varepsilon, \quad (8.15)$$

where now $\widetilde{y} = \text{id} + \varepsilon \widetilde{u}$. This choice for the recovery sequence \widehat{u}_ε corresponds to the one used in [50]. Note in particular that

$$\det \nabla \widehat{y}_\varepsilon = \det(\nabla \widetilde{y}(y_\varepsilon) \nabla y_\varepsilon) = \det \nabla \widetilde{y}(y_\varepsilon) \det \nabla y_\varepsilon > 0$$

for small ε , as $\det \nabla \widetilde{y}(y_\varepsilon) \rightarrow 1$ uniformly. That is $\mathbf{I} + \varepsilon \nabla \widehat{u}_\varepsilon \in \text{GL}^+$ almost everywhere in Ω for all ε sufficiently small. Moreover, we immediately check that

$$\widehat{u}_\varepsilon \xrightarrow{L^2} \widehat{u}_0. \quad (8.16)$$

The recovery sequence $\widehat{\mathbf{z}}_\varepsilon$ is different from the one used in [50] in two respects. At first, the choice is tailored to have a recovery sequence made of *symmetric* tensors whereas no symmetry of the recovery sequence is of course imposed in [50]. Secondly, we address here the additional intricacy of keeping the gradient of the recovery sequence bounded in L^2 while gradient terms were not discussed in [50]. Note that by neglecting \log , $\mathbf{\Pi}$, and \exp in the definition of $\widehat{\mathbf{z}}_\varepsilon$ we would retrieve the classical choice $\widehat{\mathbf{z}}_\varepsilon = \mathbf{z}_\varepsilon + \widetilde{\mathbf{z}}$ which is well-suited for quadratic energies in the linear-space setting [43]. The actual definition of $\widehat{\mathbf{z}}_\varepsilon$ is hence an adaptation of the latter to the nonlinear structure of SL_{sym}^+ .

In the following we will use the shorthand notations

$$\mathbf{C}_{p\varepsilon} := \exp(2\varepsilon \mathbf{z}_\varepsilon), \quad \widetilde{\mathbf{C}}_{p\varepsilon} := \exp(2\varepsilon(\mathbf{z}_\varepsilon + \widetilde{\mathbf{z}})), \quad \widehat{\mathbf{C}}_{p\varepsilon} := \mathbf{\Pi}(\widetilde{\mathbf{C}}_\varepsilon) = \exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon).$$

Step 2: Preliminary results. By the coercivity Lemma 8.2, we have the bound

$$\|\mathbf{z}_\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon} \|\nabla \exp(2\varepsilon \mathbf{z}_\varepsilon)\|_{L^2}^2 + \varepsilon \|\mathbf{z}_\varepsilon\|_{L^\infty} \leq c. \quad (8.17)$$

We now use the uniform Lipschitz continuity of the logarithm on the set K

$$2\varepsilon |\nabla \mathbf{z}_\varepsilon| = |\nabla \log \mathbf{C}_{p\varepsilon}| \leq c |\nabla \mathbf{C}_{p\varepsilon}|.$$

which is proved in Lemma A.1 in Appendix A. In particular, we deduce that

$$\|\nabla \mathbf{z}_\varepsilon\|_{L^2}^2 \leq c. \quad (8.18)$$

For any $\alpha \in \mathbb{R}$, by expanding the exponential $\widetilde{\mathbf{C}}_{p\varepsilon}^\alpha = \exp(2\alpha\varepsilon(\mathbf{z}_\varepsilon + \widetilde{\mathbf{z}}))$, we obtain the useful expression

$$\widetilde{\mathbf{C}}_{p\varepsilon}^\alpha = 2\alpha\varepsilon \widetilde{\mathbf{z}} + \mathbf{C}_{p\varepsilon}^\alpha + \varepsilon^2 \mathbf{L}_\varepsilon \quad (8.19)$$

where

$$\mathbf{L}_\varepsilon = \sum_{n=2}^{\infty} \frac{(2\alpha)^n \varepsilon^{n-2}}{n!} ((\mathbf{z}_\varepsilon + \widetilde{\mathbf{z}})^n - \mathbf{z}_\varepsilon^n)$$

satisfies the bound

$$\|\varepsilon \mathbf{L}_\varepsilon\|_\infty + \|\mathbf{L}_\varepsilon\|_{L^2} \leq c. \quad (8.20)$$

In fact, for all $n \geq 2$, the highest power of \mathbf{z}_ε in \mathbf{L}_ε is controlled by $c\varepsilon^{n-2}|\mathbf{z}_\varepsilon|^{n-1}$ and $\|\varepsilon \mathbf{z}_\varepsilon\|_\infty + \|\mathbf{z}_\varepsilon\|_{L^2} \leq c$. Moreover, for $\alpha = 1$, we have

$$\nabla \tilde{\mathbf{C}}_{\text{p}\varepsilon} = 2\varepsilon \nabla \tilde{\mathbf{z}} + \nabla \mathbf{C}_{\text{p}\varepsilon} + \varepsilon^2 \nabla \mathbf{L}_\varepsilon \quad (8.21)$$

and we can prove that $\|\nabla \mathbf{L}_\varepsilon\|_{L^2} \leq c$. In fact, the \mathbf{z}_ε -dependent terms in the expansion of \mathbf{L}_ε behave as $\varepsilon^{n-2}\mathbf{z}_\varepsilon^k$, with $1 \leq k \leq n-1$ for any $n \geq 2$. Correspondingly, the gradient terms fulfill

$$\|\varepsilon^{n-2} \nabla \mathbf{z}_\varepsilon^k\|_{L^2} \leq k\varepsilon^{n-2} \|(\nabla \mathbf{z}_\varepsilon) \mathbf{z}_\varepsilon^{k-1}\|_{L^2} = k\varepsilon^{n-k-1} \|(\nabla \mathbf{z}_\varepsilon)(\varepsilon \mathbf{z}_\varepsilon)^{k-1}\|_{L^2}$$

and the bound $\|\nabla \mathbf{L}_\varepsilon\|_{L^2} \leq c$ follows from $\|\varepsilon \mathbf{z}_\varepsilon\|_\infty + \|\nabla \mathbf{z}_\varepsilon\|_{L^2} \leq c$.

We next define the sets

$$K_\varepsilon := \{x \in \Omega \mid \tilde{\mathbf{C}}_{\text{p}\varepsilon} \in K\} = \{x \in \Omega \mid \hat{\mathbf{C}}_{\text{p}\varepsilon} = \tilde{\mathbf{C}}_{\text{p}\varepsilon}\}.$$

In particular, note that

$$\hat{\mathbf{z}}_\varepsilon - \tilde{\mathbf{z}} - \mathbf{z}_\varepsilon = 0 \quad \text{on } K_\varepsilon.$$

The complement of K_ε has small measure. Indeed, from (8.6) and (8.19), it follows that $\|\tilde{\mathbf{C}}_{\text{p}\varepsilon} - \mathbf{I}\|_{L^2}^2 \leq c\varepsilon^2$. Moreover, one has that $|\tilde{\mathbf{C}}_{\text{p}\varepsilon}(x) - \mathbf{I}| \geq r/\sqrt{3} - 1$ for $\tilde{\mathbf{C}}_{\text{p}\varepsilon}(x) \in \text{SL}_{\text{sym}}^+ \setminus K$, that is for $x \in \Omega \setminus K_\varepsilon$. Hence,

$$|\Omega \setminus K_\varepsilon| = \int_{\Omega \setminus K_\varepsilon} dx \leq \frac{1}{(r/\sqrt{3}-1)^2} \int_{\Omega \setminus K_\varepsilon} |\tilde{\mathbf{C}}_{\text{p}\varepsilon} - \mathbf{I}|^2 dx \leq \frac{1}{(r/\sqrt{3}-1)^2} \|\tilde{\mathbf{C}}_{\text{p}\varepsilon} - \mathbf{I}\|_{L^2}^2 \leq c\varepsilon^2.$$

The following convergences will be used in the estimate of the lim sup of the hardening terms

$$\hat{\mathbf{z}}_\varepsilon - \mathbf{z}_\varepsilon \xrightarrow{L^2} \tilde{\mathbf{z}}, \quad (8.22)$$

$$\hat{\mathbf{z}}_\varepsilon + \mathbf{z}_\varepsilon \xrightarrow{L^2} \hat{\mathbf{z}}_0 + \mathbf{z}_0. \quad (8.23)$$

Indeed, on K_ε we have $\hat{\mathbf{z}}_\varepsilon - \mathbf{z}_\varepsilon = \tilde{\mathbf{z}}$ and $\hat{\mathbf{z}}_\varepsilon + \mathbf{z}_\varepsilon = \tilde{\mathbf{z}} + 2\mathbf{z}_\varepsilon$ with $\mathbf{z}_\varepsilon \xrightarrow{L^2} \mathbf{z}_0$. Hence, convergences (8.22)-(8.23) follow from $|\Omega \setminus K_\varepsilon| < C\varepsilon^2$ and the L^2 -boundedness of $\hat{\mathbf{z}}_\varepsilon$ and \mathbf{z}_ε . In particular, by taking the sum of (8.22) and (8.23) we conclude that

$$\hat{\mathbf{z}}_\varepsilon \xrightarrow{L^2} \hat{\mathbf{z}}_0. \quad (8.24)$$

Step 3: The lim sup inequality for the dissipation. Let us decompose

$$\mathcal{D}_\varepsilon(\mathbf{z}_\varepsilon, \hat{\mathbf{z}}_\varepsilon) = \frac{1}{2\varepsilon} \int_{\Omega \setminus K_\varepsilon} D(\mathbf{C}_{\text{p}\varepsilon}, \hat{\mathbf{C}}_{\text{p}\varepsilon}) dx + \frac{1}{2\varepsilon} \int_{K_\varepsilon} D(\exp(2\varepsilon \mathbf{z}_\varepsilon), \exp(2\varepsilon \hat{\mathbf{z}}_\varepsilon)) dx.$$

Taking into account the uniform Lipschitz continuity of D on K , we have

$$\begin{aligned} \frac{1}{\varepsilon} D(\mathbf{C}_{\text{p}\varepsilon}, \widehat{\mathbf{C}}_{\text{p}\varepsilon}) &= \frac{1}{\varepsilon} D(\Pi(\mathbf{C}_{\text{p}\varepsilon}), \Pi(\widehat{\mathbf{C}}_{\text{p}\varepsilon})) \leq \frac{c}{\varepsilon} |\Pi(\mathbf{C}_{\text{p}\varepsilon}) - \Pi(\widehat{\mathbf{C}}_{\text{p}\varepsilon})| \\ &\leq \frac{c}{\varepsilon} |\mathbf{C}_{\text{p}\varepsilon} - \widehat{\mathbf{C}}_{\text{p}\varepsilon}| \stackrel{(8.19)}{=} c |2\tilde{\mathbf{z}} + \varepsilon \mathbf{L}_\varepsilon| \end{aligned}$$

and the right-hand side is uniformly bounded in L^∞ . Since $|\Omega \setminus K_\varepsilon| < c\varepsilon^2$, it follows that

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{D}_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0} \int_{K_\varepsilon} D_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \, dx.$$

On the other hand, by recalling (4.3), on the set K_ε we have that

$$\frac{1}{2\varepsilon} D(\exp(2\varepsilon \mathbf{z}_\varepsilon), \exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon)) \leq \widetilde{R}(\widehat{\mathbf{z}}_\varepsilon - \mathbf{z}_\varepsilon) = \widetilde{R}(\tilde{\mathbf{z}}),$$

hence

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{D}_\varepsilon(\mathbf{z}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \leq \int_{\Omega} R(\tilde{\mathbf{z}}) \, dx = \mathcal{D}_0(\mathbf{z}_0, \widehat{\mathbf{z}}_0).$$

Step 4: The lim sup inequality for the gradient. We aim at showing that

$$\begin{aligned} &\limsup_{\varepsilon \rightarrow 0} \frac{\mu}{2\varepsilon^2} \left(\int_{\Omega} |\nabla \exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon)|^2 \, dx - \int_{\Omega} |\nabla \exp(2\varepsilon \mathbf{z}_\varepsilon)|^2 \, dx \right) \\ &\leq 2\mu \int_{\Omega} |\nabla \widehat{\mathbf{z}}_0|^2 \, dx - 2\mu \int_{\Omega} |\nabla \mathbf{z}_0|^2 \, dx. \end{aligned} \tag{8.25}$$

By the contractive character of Π , see (8.14), we have

$$|\nabla \exp(2\varepsilon \widehat{\mathbf{z}}_\varepsilon)| = |\nabla(\Pi(\widetilde{\mathbf{C}}_{\text{p}\varepsilon}))| \leq |\nabla \widetilde{\mathbf{C}}_{\text{p}\varepsilon}|.$$

By using the decomposition (8.21) and the bound $\|\nabla \mathbf{C}_{\text{p}\varepsilon}\|_{L^2} \leq c\varepsilon$, from the coercivity condition (8.5) we compute

$$\begin{aligned} \frac{1}{\varepsilon^2} (\|\nabla \widetilde{\mathbf{C}}_{\text{p}\varepsilon}\|_{L^2}^2 - \|\nabla \mathbf{C}_{\text{p}\varepsilon}\|_{L^2}^2) &= \|2\nabla \tilde{\mathbf{z}} + \varepsilon^{-1} \nabla \mathbf{C}_{\text{p}\varepsilon} + \varepsilon \nabla \mathbf{L}_\varepsilon\|_{L^2}^2 - \|\varepsilon^{-1} \nabla \mathbf{C}_{\text{p}\varepsilon}\|_{L^2}^2 \\ &\leq 4\|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + 2\varepsilon^{-1} \|\nabla \tilde{\mathbf{z}} \nabla \mathbf{C}_{\text{p}\varepsilon} + \nabla \mathbf{C}_{\text{p}\varepsilon} \nabla \tilde{\mathbf{z}}\|_{L^1} + c\varepsilon. \end{aligned}$$

Owing to convergence (8.7) we have that $(2\varepsilon)^{-1} \nabla \mathbf{C}_{\text{p}\varepsilon} \xrightarrow{L^2} \nabla \mathbf{z}_0$, so that

$$\begin{aligned} &\limsup_{\varepsilon \rightarrow 0} \left(\frac{\mu}{2\varepsilon^2} \left(\|\nabla \widehat{\mathbf{C}}_{\text{p}\varepsilon}\|_{L^2}^2 - \|\nabla \mathbf{C}_{\text{p}\varepsilon}\|_{L^2}^2 \right) \right) \\ &\leq 2\mu \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + 2\mu \|\nabla \tilde{\mathbf{z}} \nabla \mathbf{z}_0 + \nabla \mathbf{z}_0 \nabla \tilde{\mathbf{z}}\|_{L^1} = \\ &= 2\mu \|\nabla(\tilde{\mathbf{z}} + \mathbf{z}_0)\|_{L^2}^2 - 2\mu \|\nabla \mathbf{z}_0\|_{L^2}^2 = 2\mu \|\nabla \widehat{\mathbf{z}}\|_{L^2}^2 - 2\mu \|\nabla \mathbf{z}_0\|_{L^2}^2 \end{aligned}$$

which corresponds to (8.25).

Step 5: the lim sup inequality for the elastic energies. Let \mathbf{A}_ε be defined by (8.8) and $\widehat{\mathbf{A}}_\varepsilon$ have an analogous expression in terms of \widehat{u}_ε and $\widehat{\mathbf{z}}_\varepsilon$. We aim at proving that

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \left(\int_{\Omega} W_e^\varepsilon(\widehat{\mathbf{A}}_\varepsilon) \, dx - \int_{\Omega} W_e^\varepsilon(\mathbf{A}_\varepsilon) \, dx \right) \\ & \leq \frac{1}{2} \int_{\Omega} |\nabla \widehat{u}_0^{\text{sym}} - \widehat{\mathbf{z}}_0|_{\mathbb{C}}^2 \, dx - \frac{1}{2} \int_{\Omega} |\nabla u_0^{\text{sym}} - \mathbf{z}_0|_{\mathbb{C}}^2 \, dx, \end{aligned}$$

where we have used the short-hand notation $W_e^\varepsilon(\mathbf{A}) := \varepsilon^{-2} W_e(\mathbf{I} + \varepsilon \mathbf{A})$. We preliminarily observe that

$$\|\mathbf{A}_\varepsilon\|_{L^2} + \|\widehat{\mathbf{A}}_\varepsilon\|_{L^2} \leq c. \quad (8.26)$$

Indeed, by using the decomposition (8.6) we have

$$\mathbf{C}_{p\varepsilon}^{-1/2} = \mathbf{I} + \varepsilon \mathbf{L}_\varepsilon$$

with \mathbf{L}_ε satisfying the bound (8.20). In particular, one has

$$\mathbf{A}_\varepsilon = \nabla u_\varepsilon + \mathbf{L}_\varepsilon + \nabla u_\varepsilon(\varepsilon \mathbf{L}_\varepsilon)$$

so that the bound for $\|\mathbf{A}_\varepsilon\|_{L^2} \leq c$ follows. The control of $\widehat{\mathbf{A}}_\varepsilon$ is analogous. On the set K_ε we have that

$$\widehat{\mathbf{A}}_\varepsilon = \nabla \widehat{u}_\varepsilon + \widehat{\mathbf{L}}_\varepsilon + \nabla \widehat{u}_\varepsilon(\varepsilon \widehat{\mathbf{L}}_\varepsilon)$$

for some $\widehat{\mathbf{L}}_\varepsilon$ fulfilling (8.20) and we use the fact that $|\Omega \setminus K_\varepsilon| \leq c\varepsilon^2$.

We next remark that

$$\nabla \widehat{u}_\varepsilon - \nabla u_\varepsilon \xrightarrow{L^2} \nabla \tilde{u}. \quad (8.27)$$

Indeed, by computing

$$\begin{aligned} \nabla \widehat{u}_\varepsilon &= \frac{1}{\varepsilon} (\nabla \tilde{y}(y_\varepsilon) \nabla y_\varepsilon - \mathbf{I}) = \frac{1}{\varepsilon} ((\mathbf{I} + \varepsilon \nabla \tilde{u})(y_\varepsilon) \nabla y_\varepsilon - \mathbf{I}) \\ &= \frac{1}{\varepsilon} (\nabla y_\varepsilon + \varepsilon \nabla \tilde{u}(y_\varepsilon) \nabla y_\varepsilon - \mathbf{I}) = \nabla u_\varepsilon + \nabla \tilde{u}(y_\varepsilon) + \varepsilon \nabla \tilde{u}(y_\varepsilon) \nabla u_\varepsilon \end{aligned}$$

we obtain that

$$\begin{aligned} \|(\nabla \widehat{u}_\varepsilon - \nabla u_\varepsilon) - \nabla \tilde{u}\|_{L^2} &\leq \|\nabla \tilde{u}(y_\varepsilon) - \nabla \tilde{u}\|_{L^2} + \|\varepsilon \nabla \tilde{u}(y_\varepsilon) \nabla u_\varepsilon\|_{L^2} \\ &\leq c\varepsilon + c\varepsilon \|\nabla u_\varepsilon\|_{L^2} \leq c\varepsilon. \end{aligned}$$

On the other hand, we readily check that

$$\nabla \widehat{u}_\varepsilon + \nabla u_\varepsilon = 2\nabla u_\varepsilon + \nabla \tilde{u}(y_\varepsilon) + \varepsilon \nabla \tilde{u}(y_\varepsilon) \nabla u_\varepsilon$$

so that the convergence

$$\nabla \widehat{u}_\varepsilon + \nabla u_\varepsilon \xrightarrow{L^2} \nabla \widehat{u}_0 + \nabla u_0 \quad (8.28)$$

follows. By combining (8.16) and (8.27)-(8.28) we obtain that

$$\widehat{u}_\varepsilon \xrightarrow{H^1} \widehat{u}_0. \quad (8.29)$$

As $\|\nabla \widehat{\mathbf{z}}_\varepsilon\|_{L^2}$ is bounded, convergences (8.24) and (8.29) entail that

$$(\widehat{u}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon) \xrightarrow{Q} (\widehat{u}_0, \widehat{\mathbf{z}}_0)$$

as required by Lemma 8.5.

We now turn to the proof of the two convergences

$$\widehat{\mathbf{A}}_\varepsilon - \mathbf{A}_\varepsilon \xrightarrow{L^2} \nabla \tilde{u} - \tilde{\mathbf{z}} = (\nabla \widehat{u}_0 - \widehat{\mathbf{z}}_0) - (\nabla u_0 - \mathbf{z}_0), \quad (8.30)$$

$$\widehat{\mathbf{A}}_\varepsilon + \mathbf{A}_\varepsilon \xrightarrow{L^2} (\nabla \widehat{u}_0 - \widehat{\mathbf{z}}_0) + (\nabla u_0 - \mathbf{z}_0). \quad (8.31)$$

On the set K_ε we use (8.19) for $\alpha = -1/2$ in order to get that

$$\begin{aligned} \widehat{\mathbf{A}}_\varepsilon - \mathbf{A}_\varepsilon &= \frac{1}{\varepsilon} \left((\mathbf{I} + \varepsilon \nabla \widehat{u}_\varepsilon) \widetilde{\mathbf{C}}_{\text{p}\varepsilon}^{-1/2} - (\mathbf{I} + \varepsilon \nabla u_\varepsilon) \mathbf{C}_{\text{p}\varepsilon}^{-1/2} \right) \\ &= \frac{1}{\varepsilon} \left(\widetilde{\mathbf{C}}_{\text{p}\varepsilon}^{-1/2} - \mathbf{C}_{\text{p}\varepsilon}^{-1/2} \right) + \nabla \tilde{u} \mathbf{C}_{\text{p}\varepsilon}^{-1/2} + (\nabla \widehat{u}_\varepsilon - \nabla u_\varepsilon - \nabla \tilde{u}) \mathbf{C}_{\text{p}\varepsilon}^{-1/2} + \nabla \widehat{u}_\varepsilon \left(\widetilde{\mathbf{C}}_{\text{p}\varepsilon}^{-1/2} - \mathbf{C}_{\text{p}\varepsilon}^{-1/2} \right) \\ &= (-\tilde{\mathbf{z}} + \varepsilon \mathbf{L}_\varepsilon) + \nabla \tilde{u} \mathbf{C}_{\text{p}\varepsilon}^{-1/2} + (\nabla \widehat{u}_\varepsilon - \nabla u_\varepsilon - \nabla \tilde{u}) \mathbf{C}_{\text{p}\varepsilon}^{-1/2} + \varepsilon \nabla \widehat{u}_\varepsilon (-\tilde{\mathbf{z}} + \varepsilon \mathbf{L}_\varepsilon) \end{aligned}$$

with $\|\mathbf{L}_\varepsilon\|_{L^2} \leq c$. The first term in the above right-hand side converges L^2 -strongly to $-\tilde{\mathbf{z}}$. Since $\mathbf{C}_{\text{p}\varepsilon}^{-1/2} \xrightarrow{L^2} \mathbf{I}$ by (8.6), the second term strongly converges in L^2 to $\nabla \tilde{u}$. The last two terms are easily seen to be strongly L^2 convergent to zero. Since $|\Omega \setminus K_\varepsilon| \leq c\varepsilon^2$, the bound (8.26) yields the convergence (8.30).

The proof of the weak convergence (8.31) results as a combination of the same argument for (8.9) on K_ε and the L^2 -boundedness of \mathbf{A}_ε and $\widehat{\mathbf{A}}_\varepsilon$.

We now define for all $\delta > 0$ the sets

$$U_\varepsilon^\delta := \{x \in \Omega \mid |\varepsilon \mathbf{A}_\varepsilon(x)| + |\varepsilon \widehat{\mathbf{A}}_\varepsilon(x)| \leq \tilde{c}_\delta\}$$

with \tilde{c}_δ from (5.6). On these sets, also by using the bound (8.26), we have

$$\begin{aligned} W_\varepsilon^\varepsilon(\widehat{\mathbf{A}}_\varepsilon) - W_\varepsilon^\varepsilon(\mathbf{A}_\varepsilon) &\leq (1+\delta)|\widehat{\mathbf{A}}_\varepsilon|_\mathbb{C}^2 - (1-\delta)|\mathbf{A}_\varepsilon|_\mathbb{C}^2 \leq |\widehat{\mathbf{A}}_\varepsilon|_\mathbb{C}^2 - |\mathbf{A}_\varepsilon|_\mathbb{C}^2 + 2c\delta\tilde{c}_\delta^2 = \\ &= \frac{1}{2}(\widehat{\mathbf{A}}_\varepsilon - \mathbf{A}_\varepsilon) : \mathbb{C}(\widehat{\mathbf{A}}_\varepsilon + \mathbf{A}_\varepsilon) + 2c\delta\tilde{c}_\delta^2 \quad \text{on } U_\varepsilon^\delta. \end{aligned} \quad (8.32)$$

We can easily estimate the measure of the sets U_ε^δ as follows

$$|\Omega \setminus U_\varepsilon^\delta| = \int_{\Omega \setminus U_\varepsilon^\delta} dx \leq \frac{1}{\tilde{c}_\delta^2} \int_{\Omega \setminus U_\varepsilon^\delta} (|\varepsilon \mathbf{A}_\varepsilon(x)| + |\varepsilon \widehat{\mathbf{A}}_\varepsilon(x)|) dx \leq \frac{c\varepsilon^2}{\tilde{c}_\delta^2}. \quad (8.33)$$

Thanks convergences (8.30) and (8.31), estimates (8.32) and (8.33) entail

$$\begin{aligned}
& \limsup_{\varepsilon \rightarrow 0} \int_{U_\varepsilon^\delta} \left(W_e^\varepsilon(\widehat{\mathbf{A}}_\varepsilon) - W_e^\varepsilon(\mathbf{A}_\varepsilon) \right) dx \\
& \leq \limsup_{\varepsilon \rightarrow 0} \left(2c\delta\tilde{c}_\delta^2|\Omega| + \frac{1}{2} \int_{U_\varepsilon^\delta} (\widehat{\mathbf{A}}_\varepsilon - \mathbf{A}_\varepsilon) : \mathbb{C}(\widehat{\mathbf{A}}_\varepsilon + \mathbf{A}_\varepsilon) dx \right) \\
& \leq 2c\delta\tilde{c}_\delta^2|\Omega| + \limsup_{\varepsilon \rightarrow 0} \left(\frac{1}{2} \int_{U_\varepsilon^\delta} (\nabla \tilde{u} - \tilde{\mathbf{z}}) : \mathbb{C}(\nabla \widehat{u}_0 - \widehat{\mathbf{z}}_0 + \nabla u_0 - \mathbf{z}_0) dx \right) \\
& \leq 2c\delta\tilde{c}_\delta^2|\Omega| + \frac{1}{2} \int_{\Omega} (|\nabla \widehat{u}_0^{\text{sym}} - \widehat{\mathbf{z}}_0|_{\mathbb{C}}^2 - |\nabla u_0^{\text{sym}} - \mathbf{z}_0|_{\mathbb{C}}^2) dx
\end{aligned}$$

where the minor-symmetry property $\mathbb{C}\mathbf{A} = \mathbb{C}\mathbf{A}^{\text{sym}}$ has been used.

We shall now discuss the contribution of the the sets $\Omega \setminus U_\varepsilon^\delta$. We show that the corresponding elastic-energies terms are uniformly bounded by $c\varepsilon$. Consider relation

$$\nabla \widehat{y}_\varepsilon \widehat{\mathbf{C}}_{\text{p}\varepsilon}^{-1/2} = (\nabla \widehat{y}_\varepsilon \nabla y_\varepsilon^{-1}) (\nabla y_\varepsilon \mathbf{C}_{\text{p}\varepsilon}^{-1/2}) (\mathbf{C}_{\text{p}\varepsilon}^{1/2} \widehat{\mathbf{C}}_{\text{p}\varepsilon}^{-1/2}).$$

This can be rewritten as

$$1 + \varepsilon \widehat{\mathbf{A}}_\varepsilon = \mathbf{G}_{1,\varepsilon} (1 + \varepsilon \mathbf{A}_\varepsilon) \mathbf{G}_{2,\varepsilon}$$

with

$$\mathbf{G}_{1,\varepsilon} := \nabla \widehat{y}_\varepsilon \nabla y_\varepsilon^{-1}, \quad \mathbf{G}_{2,\varepsilon} := \mathbf{C}_{\text{p}\varepsilon}^{1/2} \widehat{\mathbf{C}}_{\text{p}\varepsilon}^{-1/2}.$$

Recalling the choice (8.15), we have

$$\mathbf{G}_{1,\varepsilon} - \mathbf{I} = \nabla(\tilde{y} \circ y_\varepsilon) \nabla y_\varepsilon^{-1} - \mathbf{I} = \nabla \tilde{y} - \mathbf{I} = \varepsilon(\nabla \tilde{u}) \circ y_\varepsilon.$$

Now we consider

$$\mathbf{G}_{2,\varepsilon} - \mathbf{I} = \mathbf{C}_{\text{p}\varepsilon}^{1/2} \left((\Pi(\tilde{\mathbf{C}}_{\text{p}\varepsilon}))^{-1/2} - \mathbf{C}_{\text{p}\varepsilon}^{-1/2} \right).$$

By using the Lipschitz-continuity of the matrix square root, see Lemma A.1), we have

$$|\mathbf{G}_{2,\varepsilon} - \mathbf{I}| \leq c |\Pi(\tilde{\mathbf{C}}_{\text{p}\varepsilon}) - \mathbf{C}_{\text{p}\varepsilon}| \leq c |\tilde{\mathbf{C}}_{\text{p}\varepsilon} - \mathbf{C}_{\text{p}\varepsilon}| \leq c\varepsilon |2\tilde{\mathbf{z}} + \varepsilon \mathbf{L}_\varepsilon|.$$

The uniform bounds

$$\|\mathbf{G}_{1,\varepsilon} - \mathbf{I}\|_{L^\infty} \leq c\varepsilon, \quad \|\mathbf{G}_{2,\varepsilon} - \mathbf{I}\|_{L^\infty} \leq c\varepsilon \quad (8.34)$$

then follow. These bounds allow us to use the following estimate [50, Lemma 4.1]

$$|W_e(\mathbf{G}_1 \mathbf{F} \mathbf{G}_2) - W_e(\mathbf{F})| \leq c(1 + W_e(\mathbf{F}))(|\mathbf{G}_1 - \mathbf{I}| + |\mathbf{G}_2 - \mathbf{I}|) \quad \forall |\mathbf{G}_1|, |\mathbf{G}_2| \leq \delta \quad (8.35)$$

for some constants $c, \delta > 0$. By combining this with the bounds (8.34) one has that

$$\begin{aligned}
& \int_{\Omega \setminus U_\varepsilon^\delta} (W_e^\varepsilon(\widehat{\mathbf{A}}_\varepsilon) - W_e^\varepsilon(\mathbf{A}_\varepsilon)) dx = \frac{1}{\varepsilon^2} \int_{\Omega \setminus U_\varepsilon^\delta} (W_e(\mathbf{G}_{1,\varepsilon} \mathbf{F}_\varepsilon \mathbf{G}_{1,\varepsilon}) - W_e(\mathbf{F}_\varepsilon)) dx \\
& \stackrel{(8.35)}{\leq} \frac{c}{\varepsilon^2} \int_{\Omega \setminus U_\varepsilon^\delta} (1 + W_e(\mathbf{F}_\varepsilon)) (|\mathbf{G}_1 - \mathbf{I}| + |\mathbf{G}_2 - \mathbf{I}|) dx \\
& \stackrel{(8.34)}{\leq} \frac{c}{\varepsilon} \int_{\Omega \setminus U_\varepsilon^\delta} (1 + W_e(\mathbf{F}_\varepsilon)) dx \stackrel{(8.33)}{\leq} c\varepsilon,
\end{aligned}$$

which completes the proof.

Step 6: The lim sup inequality for the plastic energy. We aim at showing that

$$\limsup_{\varepsilon \rightarrow 0} \left(\int_{\Omega} W_p^\varepsilon(\widehat{\mathbf{z}}_\varepsilon) dx - \int_{\Omega} W_p^\varepsilon(\mathbf{z}_\varepsilon) dx \right) \leq \frac{1}{2} \int_{\Omega} |\widehat{\mathbf{z}}_0|_{\mathbb{H}}^2 dx - \frac{1}{2} \int_{\Omega} |\mathbf{z}_0|_{\mathbb{H}}^2 dx, \quad (8.36)$$

where we have used the short-hand notation $W_p^\varepsilon(\mathbf{z}) := \varepsilon^{-2} W_p(\exp(2\varepsilon \mathbf{z}))$. The strategy is similar to the one used for the elastic energy. First, we define the sets

$$Z_\varepsilon^\delta := \{x \in \Omega \mid |\varepsilon \mathbf{z}_\varepsilon(x)| + |\varepsilon \widehat{\mathbf{z}}_\varepsilon(x)| \leq c_\delta\},$$

with c_δ from (5.4), so that

$$W_p^\varepsilon(\widehat{\mathbf{z}}_\varepsilon) - W_p^\varepsilon(\mathbf{z}_\varepsilon) \leq \frac{1}{2} (\widehat{\mathbf{z}}_\varepsilon - \mathbf{z}_\varepsilon) : \mathbb{H}(\widehat{\mathbf{z}}_\varepsilon + \mathbf{z}_\varepsilon) + 2c_\delta c_\delta^2 \quad \text{on } Z_\varepsilon^\delta. \quad (8.37)$$

Arguing exactly as in Step 5, we can prove that the complementary sets $\Omega \setminus Z_\varepsilon^\delta$ fulfill

$$|\Omega \setminus Z_\varepsilon^\delta| \leq \frac{c\varepsilon^2}{c_\delta^2}.$$

Owing to the Lipschitz-continuity of W_p on K , the contraction property of $\mathbf{\Pi}$ and (8.19)

$$\begin{aligned} \int_{\Omega \setminus Z_\varepsilon^\delta} (W_p^\varepsilon(\widehat{\mathbf{z}}_\varepsilon) - W_p^\varepsilon(\mathbf{z}_\varepsilon)) dx &= \frac{1}{\varepsilon^2} \int_{\Omega \setminus Z_\varepsilon^\delta} (\widehat{W}_p(\widehat{\mathbf{C}}_{p\varepsilon}) - \widehat{W}_p(\mathbf{C}_{p\varepsilon})) dx \leq \\ \frac{c}{\varepsilon^2} \int_{\Omega \setminus Z_\varepsilon^\delta} |\widehat{\mathbf{C}}_{p\varepsilon} - \mathbf{C}_{p\varepsilon}| dx &\leq \frac{c}{\varepsilon^2} \int_{\Omega \setminus Z_\varepsilon^\delta} |\widetilde{\mathbf{C}}_{p\varepsilon} - \mathbf{C}_{p\varepsilon}| dx \leq \frac{c}{\varepsilon^2} |\Omega \setminus Z_\varepsilon^\delta| \varepsilon = c\varepsilon. \end{aligned} \quad (8.38)$$

By combining (8.37)-(8.38) and (8.22)-(8.23), the lim sup condition (8.36) follows from δ being arbitrary. \square

Proof of Theorem 8.1. Having established Lemmas 8.3, 8.3, and 8.5, we are in the position of applying the abstract Lemma 3.2. Although Lemma 8.5 deals with smooth and compactly supported competitors only, note that the full strength of condition (3.9) can be easily recovered by density.

The pointwise strong convergence of $(u_\varepsilon, \mathbf{z}_\varepsilon)$ and the convergence of energies and dissipation follow at once from the uniform convexity of the linearized energy \mathcal{E}_0 along the same lines as [50, Cor. 3.8 and Cor. 3.9]. \square

APPENDIX A. LOCAL LIPSCHITZ CONTINUITY

We comment here on the local Lipschitz continuity of the matrix logarithm and the matrix fractional power on SL_{sym}^+ . This is a consequence of the unit determinant constraint, which allows to control the moduli of the matrix eigenvalues and their reciprocals in terms of the matrix norm.

Lemma A.1 (Local Lipschitz continuity). *We have that*

$$|\log \mathbf{C}_1 - \log \mathbf{C}_2| \leq c(1 + (|\mathbf{C}_1| \vee |\mathbf{C}_2|)^2) |\mathbf{C}_1 - \mathbf{C}_2| \quad \forall \mathbf{C}_1, \mathbf{C}_2 \in \text{SL}_{\text{sym}}^+ \quad (\text{A.1})$$

for some positive constant $c > 0$. In particular, given any compact $K \subset \text{SL}_{\text{sym}}^+$ there exists $c_K > 0$ such that $|\log \mathbf{C}_1 - \log \mathbf{C}_2| \leq c_K |\mathbf{C}_1 - \mathbf{C}_2|$ for all $\mathbf{C}_1, \mathbf{C}_2 \in K$. Moreover, for all $\alpha \in \mathbb{R}$, we have that

$$|\mathbf{C}_1^\alpha - \mathbf{C}_2^\alpha| \leq c_{K\alpha} |\mathbf{C}_1 - \mathbf{C}_2| \quad \forall \mathbf{C}_1, \mathbf{C}_2 \in K \quad (\text{A.2})$$

for some positive constant $c_{K\alpha}$.

Proof. Let $\sigma_i \subset (0, \infty)$ be the spectrum of \mathbf{C}_i , for $i = 1, 2$, and $\lambda_0 = \min\{\sigma_1 \cup \sigma_2\} > 0$. Since $\det \mathbf{C}_i = 1$ one easily checks that

$$\lambda_0 \geq |\mathbf{C}_1|^{-2} \wedge |\mathbf{C}_2|^{-2}. \quad (\text{A.3})$$

The logarithm of \mathbf{C}_i can be calculated via the Cauchy Integral Formula (for operators) [19, Ch. 7]

$$\log \mathbf{C}_i = \int_{\gamma} \frac{\log z}{z\mathbf{I} - \mathbf{C}_i} dz,$$

where γ is a closed contour in the analyticity region of $\log z$ (one can take $\gamma \subset \{\text{Re } z > 0\}$, for instance) and winds one time around $\sigma_1 \cup \sigma_2$. Therefore

$$\begin{aligned} \log \mathbf{C}_1 - \log \mathbf{C}_2 &= (\mathbf{C}_1 - \mathbf{C}_2) \int_{\gamma} \frac{\log z}{(z\mathbf{I} - \mathbf{C}_1)(z\mathbf{I} - \mathbf{C}_2)} dz \\ &= (\mathbf{C}_1 - \mathbf{C}_2) \int_{\bar{\gamma}} \frac{\log z}{(z\mathbf{I} - \mathbf{C}_1)(z\mathbf{I} - \mathbf{C}_2)} dz, \end{aligned}$$

where, in the last equality, we have replaced γ the infinite straight line $\bar{\gamma} = \{x_0 + it \mid t \in \mathbb{R}\}$, $x_0 \in (0, \lambda_0)$, since the modulus of the integrand behaves like $z \mapsto |\log z| |z|^{-2}$ at infinity. For all $z \in \bar{\gamma}$ we have

$$\text{Re } z = x_0 < \lambda_0 \Rightarrow \left| \frac{1}{z\mathbf{I} - \mathbf{C}_i} \right| \leq \frac{\sqrt{3}}{|z - \lambda_0|}.$$

We hence compute that

$$|\log \mathbf{C}_1 - \log \mathbf{C}_2| \leq |\mathbf{C}_1 - \mathbf{C}_2| \int_{\bar{\gamma}} \frac{3|\log z|}{|z - \lambda_0|^2} dz \leq \frac{3}{2} |\mathbf{C}_1 - \mathbf{C}_2| \int_{-\infty}^{\infty} \frac{|\log(x_0^2 + t^2)| + \pi}{(x_0 - \lambda_0)^2 + t^2} dt.$$

The last inequality follows from the elementary control

$$|\log(x_0 + it)| \leq \frac{1}{2} |\log(x_0^2 + t^2)| + |\vartheta| \leq \frac{1}{2} (|\log(x_0^2 + t^2)| + \pi)$$

for $\vartheta := \arctan(t/x_0) \in (-\pi/2, \pi/2)$. As this estimate holds for any $x_0 \in (0, \lambda_0)$, by letting $x_0 \rightarrow 0$ we obtain

$$|\log \mathbf{C}_1 - \log \mathbf{C}_2| \leq 3 |\mathbf{C}_1 - \mathbf{C}_2| \int_0^{\infty} \frac{|\log t^2| + \pi}{\lambda_0^2 + t^2} dt.$$

We can now elementarily compute that

$$\int_0^\infty \frac{\pi}{\lambda_0^2 + t^2} dt = \frac{\pi^2}{2\lambda_0}, \quad \int_0^\infty \frac{|\log t^2|}{\lambda_0^2 + t^2} dt \leq -2 \int_0^1 \frac{\log t}{\lambda_0^2} dt + \int_1^\infty \frac{\log t^2}{t^2} dt = \frac{2}{\lambda_0^2} + c.$$

Eventually, we have proved that

$$|\log \mathbf{C}_1 - \log \mathbf{C}_2| \leq c \left(1 + \frac{1}{\lambda_0^2}\right) |\mathbf{C}_1 - \mathbf{C}_2| \stackrel{(A.3)}{\leq} c (1 + (|\mathbf{C}_1| \vee |\mathbf{C}_2|)^2) |\mathbf{C}_1 - \mathbf{C}_2|.$$

As for the matrix power $\mathbf{C} \mapsto \mathbf{C}^\alpha$, we simply use $\mathbf{C}^\alpha = \exp(\alpha \log \mathbf{C})$ and recall that the exponential map is uniformly Lipschitz on compact sets. \square

APPENDIX B. THE MAP Π

We collect here some comment on the existence of a map

$$\Pi : \text{SL}_{\text{sym}}^+ \rightarrow K$$

having properties (8.14), to be used in the definition of the recovery sequence (8.13). Recall that

$$K = \{\mathbf{C} \in \text{SL}_{\text{sym}}^+ \mid |\mathbf{C}| \leq r\}, \quad r > |\mathbf{I}| = \sqrt{3}$$

and let the flux Φ_t , $t \geq 0$, be associated to the following differential equation on GL^+

$$\dot{\mathbf{C}} = -(\mathbf{C} - 3|\mathbf{C}^{-1}|^{-2}\mathbf{C}^{-1}). \quad (\text{B.1})$$

In particular, $t \mapsto \Phi_t(\mathbf{C})$ is the solution of the differential equation (B.1) with initial datum \mathbf{C} . Note that the manifold SL_{sym}^+ is invariant under the flux Φ_t . In fact, along solutions $\mathbf{C}(t)$ of the equation (B.1) symmetry and determinant constraint are preserved as the symmetry of \mathbf{C} induces that of $\dot{\mathbf{C}}$ and

$$\text{tr}(\mathbf{C}(t)^{-1}\dot{\mathbf{C}}(t)) = -\text{tr}(\mathbf{I} - 3|\mathbf{C}^{-1}|^{-2}\mathbf{C}^{-2}) = -(3 - 3|\mathbf{C}^{-1}|^{-2}\mathbf{C}^{-2}:\mathbf{I}) = 0.$$

Moreover, the flux Φ_t is norm-contractive for we readily check that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\mathbf{C}(t)|^2 &= \text{tr}(\mathbf{C}\dot{\mathbf{C}}) = -\text{tr}(\mathbf{C}^2 - 3|\mathbf{C}^{-1}|^{-2}\mathbf{I}) \\ &= -(|\mathbf{C}|^2 - 9|\mathbf{C}^{-1}|^{-2}) \leq 3 - |\mathbf{C}|^2 \leq 0. \end{aligned}$$

We have here used the fact that $\mathbf{C}^{-1} \in \text{SL}_{\text{sym}}^+$ and $|\mathbf{C}^{-1}|^2 \geq 3$. More precisely, as $|\mathbf{C}|^2 \geq 3$ on SL_{sym}^+ (with equality corresponding to $\mathbf{C} = \mathbf{I}$), we have checked that

$$|\mathbf{C}| > \sqrt{3} \Rightarrow \frac{1}{2} \frac{d}{dt} |\mathbf{C}(t)|^2 < 0. \quad (\text{B.2})$$

Let us record some additional properties of the flux Φ_t in the following lemma.

Lemma B.1. *The flux Φ_t satisfies the following properties*

i) Let $\mathbf{C}, \mathbf{C}_0 \in \text{SL}_{\text{sym}}^+$. Then

$$|\Phi_t(\mathbf{C})| \geq |\mathbf{C}_0| \Rightarrow \frac{d}{dt} |\Phi_t(\mathbf{C}) - \mathbf{C}_0| \leq 0. \quad (\text{B.3})$$

ii) For all $t \geq 0$, Φ_t is a contraction on SL_{sym}^+ , namely

$$|\Phi_t(\mathbf{C}_1) - \Phi_t(\mathbf{C}_2)| \leq |\mathbf{C}_1 - \mathbf{C}_2| \quad \forall \mathbf{C}_1, \mathbf{C}_2 \in \text{SL}_{\text{sym}}^+. \quad (\text{B.4})$$

Proof. Ad i). Let $\mathbf{C}(t) = \Phi_t(\mathbf{C})$ for some $\mathbf{C} \in \text{SL}_{\text{sym}}^+$. The differential equation (B.1) entails that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\mathbf{C}(t) - \mathbf{C}_0|^2 &= \text{tr} \left(\dot{\mathbf{C}}(\mathbf{C} - \mathbf{C}_0) \right) \\ &= -\text{tr}(\mathbf{C}(\mathbf{C} - \mathbf{C}_0)) + \frac{3}{|\mathbf{C}^{-1}|^2} (3 - \text{tr}(\mathbf{C}^{-1}\mathbf{C}_0)) \end{aligned}$$

By the invariance of the trace by cyclic permutation of the factors we have that

$$\text{tr}(\mathbf{C}^{-1}\mathbf{C}_0) = \text{tr}(\mathbf{C}^{-1/2}\mathbf{C}_0\mathbf{C}^{-1/2}) \geq 3$$

since $\mathbf{C}^{-1/2}\mathbf{C}_0\mathbf{C}^{-1/2} \in \text{SL}_{\text{sym}}^+$. Moreover,

$$|\mathbf{C}| \geq |\mathbf{C}_0| \Rightarrow |\mathbf{C}|^2 = \text{tr}(\mathbf{C}^2) \geq \text{tr}(\mathbf{C}\mathbf{C}_0),$$

which proves the statement.

Ad ii). Let $\mathbf{C}_i(t) = \Phi_t(\mathbf{C}_i)$ for $\mathbf{C}_1, \mathbf{C}_2 \in \text{SL}_{\text{sym}}^+$. By using again the differential equation we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\mathbf{C}_1(t) - \mathbf{C}_2(t)|^2 &= \text{tr} \left[(\dot{\mathbf{C}}_1 - \dot{\mathbf{C}}_2)(\mathbf{C}_1 - \mathbf{C}_2) \right] = \\ &= -\text{tr} \left[\left(\mathbf{C}_1 - \frac{3}{|\mathbf{C}_1^{-1}|^2} \mathbf{C}_1^{-1} - \mathbf{C}_2 + \frac{3}{|\mathbf{C}_2^{-1}|^2} \mathbf{C}_2^{-1} \right) (\mathbf{C}_1 - \mathbf{C}_2) \right] = \\ &= -|\mathbf{C}_1 - \mathbf{C}_2|^2 + 3 \left(\frac{3 - \text{tr}(\mathbf{C}_1^{-1}\mathbf{C}_2)}{|\mathbf{C}_1^{-1}|^2} + \frac{3 - \text{tr}(\mathbf{C}_2^{-1}\mathbf{C}_1)}{|\mathbf{C}_2^{-1}|^2} \right) \leq 0 \end{aligned}$$

proving the assertion. \square

For any initial datum $\mathbf{C} \in \text{SL}_{\text{sym}}^+$, by (B.2) there exists a minimum time $t_0 \geq 0$ for which $|\Phi_{t_0}(\mathbf{C})| = r > \sqrt{3}$, namely

$$t_0(\mathbf{C}) = \min\{t \geq 0 \mid \Phi_t(\mathbf{C}) \in K\} < \infty. \quad (\text{B.5})$$

We define the map $\mathbf{\Pi} : \text{SL}_{\text{sym}}^+ \rightarrow K$ as follows

$$\mathbf{\Pi}(\mathbf{C}) := \Phi_{t_0(\mathbf{C})}(\mathbf{C}). \quad (\text{B.6})$$

Of course, $\mathbf{\Pi}|_K = \text{id}$, so that the first condition in (8.14) is satisfied.

Let $\mathbf{C}_1, \mathbf{C}_2 \in \text{SL}_{\text{sym}}^+$ be given and assume with no loss of generality that $t_1 := t_0(\mathbf{C}_1) \leq t_0(\mathbf{C}_2) =: t_1 + \delta$. We can write

$$\begin{aligned} |\mathbf{\Pi}(\mathbf{C}_1) - \mathbf{\Pi}(\mathbf{C}_2)| &= |\Phi_{t_1}(\mathbf{C}_1) - \Phi_{t_2}(\mathbf{C}_2)| = |\Phi_{t_1}(\mathbf{C}_1) - \Phi_\delta(\Phi_{t_1}(\mathbf{C}_2))| \\ &\stackrel{(\text{B.3})}{\leq} |\Phi_{t_1}(\mathbf{C}_1) - \Phi_{t_1}(\mathbf{C}_2)| \stackrel{(\text{B.4})}{\leq} |\mathbf{C}_1 - \mathbf{C}_2|. \end{aligned}$$

The map $\mathbf{\Pi}$ is hence contractive in SL_{sym}^+ , which is the second condition in (8.14).

APPENDIX C. LOWER SEMICONTINUITY TOOL

For the sake of completeness, we report here the lower-semicontinuity tool which has been repeatedly used above. The lemma is in the spirit of [3, Thm. 1] and [33]. A proof can be found in [66, Thm 4.3, Cor. 4.4], in [45, Lemma 3.1] in one dimension, and in [35] in case of local uniform convergence.

Lemma C.1 (Lower-semicontinuity tool). *Let $f_0, f_\varepsilon : \mathbb{R}^n \rightarrow [0, \infty]$ be lower semicontinuous,*

$$f_0(v_0) \leq \inf \left\{ \liminf_{\varepsilon \rightarrow 0} f_\varepsilon(v_\varepsilon) \mid v_\varepsilon \rightarrow v_0 \right\} \quad \forall v_0 \in \mathbb{R}^n$$

and $w_\varepsilon \rightharpoonup w_0$ in $L^1(\Omega; \mathbb{R}^n)$. By denoting by ζ the Young measure generated by w_ε we have that

$$\int_{\Omega} \left(\int_{\mathbb{R}^n} f_0(w) d\zeta_x(w) \right) dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} f_\varepsilon(w_\varepsilon) dx.$$

In particular, if f_0 is convex we have

$$\int_{\Omega} f_0(w_0) dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} f_\varepsilon(w_\varepsilon) dx.$$

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STATEMENT ON CONFLICT OF INTERESTS

The authors declare that they have no conflict of interests.

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